# PARTITIONS INTO POWERS OF AN ALGEBRAIC NUMBER

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Integer partitions are the central object of study in additive number theory. For a fixed subset $S$ of the positive integers $\mathbb{N}$, one can investigate the properties of partitions of a given positive integer $n$ into parts from $S$. When $S$ is the set of non-negative powers of a fixed positive integer $b$, we obtain a class of partitions called the *$b$-ary partitions*.

A straightforward generalization is to fix a real number $\beta$ and consider partitions of the form

$$\alpha = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0, \tag{1}$$

where $\alpha$ is a real number. This notion is closely related to that of *$\beta$-expansions*, except that we do not take the "digits" $a_i$ from a fixed finite alphabet but allow them to be arbitrary non-negative integers.

Let $p_\beta$ denote the *partition function* associated with $\beta$, so that by definition, $p_\beta(\alpha)$ is the number of partitions of $\alpha$ into powers of $\beta$ as in (1). If we want to study the properties of this function for a given $\beta$, it is natural to require that $p_\beta(\alpha)$ should be finite for every $\alpha$ in $\mathbb{R}$. A sufficient condition for this to be the case is clearly $\beta > 1$, but this condition is not necessary. In fact, it is easy to observe that for $\beta$ transcendental, $p_\beta(\alpha)$ is always 0 or 1. When $\beta$ is an algebraic number, it is sufficient for one of its conjugates to be greater than 1. Our first theorem shows that this condition is also necessary in the case when $\beta$ is of degree two.

**Theorem 1.** *Let $\beta$ be a real root of a quadratic polynomial $Ax^2 + Bx + C$. Then $p_\beta(\alpha)$ is finite for every real $\alpha$ if and only if at least one of the conjugates $\beta$ and $\beta'$ is greater than* 1.

If $\beta$ satisfies the condition from Theorem 1, what can we say about the range of the function $p_\beta$? The answer, under some additional constraints, is provided by our second theorem. We denote by $\operatorname{Tr}\beta$ and $\operatorname{N}\beta$ the trace and norm of $\beta$, respectively. Recall that a totally real algebraic integer is called *totally positive* if all its conjugates are positive.

**Theorem 2.** *If a totally positive quadratic integer $\beta$ satisfies*

$$\operatorname{Tr}\beta \le \operatorname{N}\beta < 2\operatorname{Tr}\beta, \tag{2}$$

*then for every integer $n \ge 0$,*

$$p_\beta((\operatorname{Tr}\beta)\beta^n) = n + 1. \tag{3}$$

Let $K = \mathbb{Q}(\sqrt{D})$, where $D > 0$ is a square-free integer. It is not difficult to show that there exist infinitely many totally positive integral elements $\beta$ of $K$ satisfying (2). Therefore, we immediately obtain the following corollary.

**Corollary.** *In a real quadratic field $K = \mathbb{Q}(\sqrt{D})$, there exist infinitely many $\beta$ such that $p_\beta$ attains all non-negative integer values.*

Next, we discuss our results in a broader context. The asymptotic behavior of the function $p_b(n)$ for a rational integer $b \geq 2$ was investigated by Mahler [Ma], who proved the asymptotic equality

$$\log p_b(n) \sim \frac{(\log n)^2}{2 \log b}. \tag{4}$$

An analogous problem for an arbitrary real $\beta > 1$ was considered by de Bruijn [Br], whose work was further improved by Pennington [Pe]. If one defines $P_\beta(x)$ as the number of solutions of the inequality

$$a_n \beta^n + a_{n-1}\beta^n + \cdots + a_1\beta + a_0 < x \tag{5}$$

in non-negative integers, then

$$\log\left(P_\beta(x) - P_\beta(x-1)\right) \sim \log P_\beta(x) \sim \frac{(\log x)^2}{2\log\beta}. \tag{6}$$

It follows from (4) that $p_b(n)$ grows roughly as $\exp((\log n)^2)$, faster than any polynomial, and its range is therefore a set of density zero. In view of this, Theorem 2 and its corollary seem quite surprising.

The function $p_b$ satisfies the recurrence relations

$$p_b(nb) = p_b(nb+1) = \cdots = p_b(nb+(b-1)), \qquad p_b(nb) = p_b((n-1)b) + p_b(n), \tag{7}$$

which easily generalize to our setting:

$$p_\beta(\alpha) = \begin{cases} p_\beta(\alpha-1) + p_\beta(\alpha/\beta), & \text{if } \beta \text{ divides } \alpha \text{ in } \mathbb{Z}[\beta], \\ p_\beta(\alpha-1), & \text{otherwise.} \end{cases} \tag{8}$$

Some congruence properties of the binary partition function were discovered by Churchhouse [Ch], while the case of an arbitrary integer $b \geq 2$ was studied by Żmija [Zm]. Let us explicitly mention only one of these results: If $n = a_0 + a_1\beta + \cdots + a_s\beta^s$ is the expansion of $n$ in base $b$, then

$$p_b(bn) \equiv \prod_{j=0}^{s}(a_j+1) \pmod{b}. \tag{9}$$

No similar identity is known when $\beta$ is not an integer. It appears that qualitative properties of partitions into powers of an algebraic number $\beta$ have not been treated anywhere in literature.

## References


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