ONE-PARAMETER DEFORMATIONS OF BOWEN-SERIES FUNCTIONS ASSOCIATED TO COCOMPACT FUCHSIAN TRIANGLE GROUPS

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The following is joint work with my Ph.D. advisor, Thomas A. Schmidt [1].

In 1979, Rufus Bowen and Caroline Series [2] claimed to define, for each signature of finitely generated Fuchsian groups of the first kind, a function on $\mathbb{S}^1$ which is similar to the standard Gauss map in that the function is expanding, Markov, transitive and satisfies Renyi's condition. Such maps admit a unique ergodic invariant measure equivalent to Lebesgue measure.

Katok and Ugarcovici [3] studied Bowen-Series functions associated to the cocompact torsion-free Fuchsian groups. They gave a one-dimensional family of functions for each of these Bowen-Series functions, and gave explicit expressions for their invariant ergodic measures. Los [6] also studied these functions in the surface group setting.

We study the Bowen-Series function f associated to a cocompact Fuchsian hyperbolic triangle group Γ. We give a correction to the paper [2] and then answer the following questions:

(Q) Can we define a family of expansive functions via an '$\alpha-$deformation' of f? If so, is there an ergodic invariant measure for each function in the family?

Bowen and Series' approach was to construct a special fundamental domain per group signature. Fundamental to the definition of their functions was that these domains satisfy an *"extension property"*. To define the property, suppose $\mathcal{F}$ is a convex fundamental domain for some Fuchsian group Γ. Let S be the set of sides of $\mathcal{F}$ and for each $s \in S$, let $g(s)$ be the geodesic containing s. Following [5], we say that $\mathcal{F}$ satisfies the *extension property* if

$$g(s) \cap \bigcup_{T \in \Gamma} T(\mathcal{F}^\circ) = \emptyset, \ \forall s \in S.$$

We have found examples where their construction fails to satisfy this property. In fact, we prove the following lemma:

Lemma. *Suppose that (m_1, m_2, m_3) is the signature of a cocompact hyperbolic Fuchsian triangle group. If more than one m_i is odd, then no convex fundamental domain for the signature has the extension property. Otherwise, the Bowen-Series fundamental domain for this signature does have the extension property.*

We thus restrict to signatures in accordance with this correction. The Bowen-Series function f is defined in terms of the side pairing elements of the Bowen-Series fundamental domain, T_i, $1 \leq i \leq 4$. There are four corresponding 'overlap intervals' $\mathscr{O}_i$ where appropriately replacing $f(x) = T_i x$ by $x \mapsto T_{i-1}x$ for all x in some subinterval determined by a parameter α leads to an eventually expansive function f_α. Hence, we answer the first part of our question and define a function family $\{f_\alpha\}$ where α runs through the set of the overlap intervals. These families are similar to those of [3, 6] in that the overlap intervals are naturally related to the geometry of the fundamental domain and allow for our function definition to guarantee expansiveness.

By Adler-Flatto's 'Folklore theorem' in [5], each of these expansive functions has a unique ergodic probability measure equivalent to Lebesgue measure if the function is (finite) Markov and has an appropriate transitivity property. Thus, we prove the following:

Theorem 1. *The map f_α is Markov if and only if α is a hyperbolic fixed point of Γ.*

For signature (m_1, m_2, m_3), we set $(n_1, n_2, n_3, n_4) = (m_3, m_2/2, m_3, m_1/2)$.

Theorem 2. *Fix $\alpha \in \mathscr{O}_i$. The function f_α is surjective if and only if the following conditions hold:*

- $n_i > 2$,
- $n_i = 2$ *and* $n_{i+2} > 2$,
- α *belongs to the closure of the set of points* $x \in \mathscr{O}_i$ *such that* $f^{n_i}(x) = f^{n_i - 1}(T_{i-1}x)$.

Moreover, if f_α is Markov, then f_α is transitive if and only if f_α is surjective.

However, the surjectivity is not always satisfied. Indeed, we will present an explicit example of a Markov f_α exhibiting this failure. Moreover, this function f_α has no invariant probability measure equivalent to Lebesgue! That is, the answer to the second part of our question is: No. We found this unexpected, given the results on similar families of one-parameter families of continued fraction-like maps.

References

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