Varying the base of numeration in the Rényi numeration dynamical system

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Abstract.
In this work we consider the Rényi numeration dynamical system $([0,1], T_\beta)$ and the β-transformation T_β for $\beta > 1$ tending to 1^+, in particular when the base of numeration β runs over the set of reciprocal algebraic integers.

Using the dynamical zeta function $\zeta_\beta(z)$ of the β-transformation, we show how an accumulation of Galois conjugates of β occurs in the neighbourhood of the unit circle in $\mathbb{C}$, when β tends to one. This phenomenon has a deep impact on the problem of universal minoration of the Mahler measure of nonzero algebraic integers which are not roots of unity.

The present work constitutes an attack of the Conjecture of Lehmer by the dynamical zeta function of the β-shift to prove that this Conjecture is true

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without any ingredient of arithmetic geometry.

In 1933 Lehmer asked the question about the existence of integer polynomials having a Mahler measure different of one, smaller than Lehmer's number $1.176280\ldots$ (and arbitrarily close to one). This problem, called Problem of Lehmer, became a Conjecture, stating that there exists a universal lower bound > 1 to the Mahler measures of the nonzero algebraic integers which are not roots of unity. So to say there would exist a gap between 1 and $1 + c$ with $c > 0$ a constant which is universal. The problem of the minoration of the Mahler measure of algebraic integers is a very deep one and has been extended to the theory of heights in arithmetic geometry (elliptic curves, Abelian varieties) by D. Masser.

We show why such a gap exists and its origin. The main arguments of the proof arise from the lenticular poles of the dynamical zeta functions $\zeta_\beta(z)$ of the Rényi–Parry arithmetical dynamical ("β-shift"), with $\beta > 1$ any real number tending to one, to which a lenticular measure can be associated, satisfying a Dobrowolski-type inequality with the dynamical degree of β. When β runs over the set of nonzero reciprocal algebraic integers, under some assumptions, the lenticular poles are identified with conjugates of β, using Kala-Vávra's periodic representation theorem (2019), and this lenticular measure is identified with a minorant of the Mahler measure of β.

Though expressed as hypergeometric functions (Mellin, 1915) the lenticularity of the poles only appears when using their (Poincaré) asymptotic expansions, as functions of the dynamical

degree, in the angular sector guessed by M. Langevin, G. Rhin and C. Smyth, G. Rhin and Q. Wu.

Whether Lehmer's number is the smallest Mahler measure > 1 of algebraic integers remains open.

En 1933 D. Lehmer posa la question de l'existence de polynômes $\in \mathbb{Z}[X]$ ayant une mesure de Mahler > 1, plus petite que le nombre de Lehmer $1.17...$ (et de valeur arbitrairement proche de 1). Sa stratégie était de construire de très grands nombres premiers à partir de tels polynômes, d'autant plus grands que la mesure de Mahler des polynômes est petite, et telle que > 1.

Le Problème de Lehmer, qui est un problème de minoration de la mesure de Mahler, est devenu la Conjecture de Lehmer ; celle-ci stipule qu'il existe une borne inférieure universelle > 1 pour toute mesure de Mahler d'entiers algébriques non nuls qui ne sont pas racines de l'unité. Autrement dit il existerait un gap entre 1 et $1+c$ pour une constante $c > 0$ universelle. On rappellera les différents cas connus de nombres algébriques pour lesquels la Conjecture est vraie.

Le présent travail constitue une attaque de la Conjecture de Lehmer par la fonction zêta dynamique du β-shift (de la β-transformation) pour démontrer que cette Conjecture est vraie, pour tous les autres cas non connus. On montre l'origine de ce gap, et l'on dit pourquoi il existe. On montre les différents ingrédients qui amènent à cette conclusion : en particulier l'existence de pôles lenticulaires dans un secteur angulaire deviné par M. Langevin ('85, '86, '88), G. Rhin and C. Smyth ('95), puis G. Rhin and Q. Wu, qui, identifiés à des conjugués de Galois par un théorème de représentation de Kala-Vavra (2019), donnent lieu à une inégalité de type Dobrowolski.

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