ADDITIVE STRUCTURE OF NON-MONOGENIC SIMPLEST CUBIC FIELDS

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(JOINT WORK WITH DANIEL GIL MUÑOZ)

The simplest cubic fields $K = \mathbb{Q}(\rho)$ were introduced by Shanks in [11]. There are generated by the root $\rho > 1$ of the polynomial $x^3 - (a+3)x^2 - ax - 1$ where a is an integer coefficient greater than -1. Since these fields have many useful properties (they are cyclic, monogenic for positive density of coefficients a, have units of all signatures, etc.), they can serve as a good starting point for the study of the totally real number fields of degrees greater than 2.

As indicated before, the ring of algebraic integers $\mathcal{O}_K$ is simply $\mathbb{Z}[\rho]$ for infinitely many coefficients a, e.g., if the square root a^2+3a+9 of the discriminant of K is square-free (but not only in these cases). The conditions under which K is monogenic are fully described in [7]. However, in this talk, we will focus on the opposite case. Moreover, we will restrict to the subfamily of non-monogenic simplest cubic fields with the integral basis of the form

$$\left\{1, \rho, \frac{k + l\rho + \rho^2}{p}\right\} \tag{1}$$

where p is a prime and $1 \leq k, l \leq p-1$ are integers such that $\frac{k+l\rho+\rho^2}{p}$ is an algebraic integer. Moreover, we will discuss for which primes p such fields exist and determine the corresponding values of k and l.

Then, we will study the so-called indecomposable integers in these fields. Let $\mathcal{O}_K^+$ be the set of totally positive elements in $\mathcal{O}_K$, i.e., of those elements which have all conjugates positive. We say that $\alpha \in \mathcal{O}_K^+$ is an indecomposable integer in $\mathcal{O}_K$ if one cannot express it as $\alpha = \beta + \gamma$ where $\beta, \gamma \in \mathcal{O}_K^+$. Although indecomposable integers play an important role in the study of universal quadratic forms [1, 5, 12, 14], we currently do not know much about these elements. Their structure was determined only for real quadratic fields [3, 10] and several families of monogenic cubic fields [6, 13]. Moreover, we have some partial results in the case of real biquadratic fields [4, 8].

In this talk, we show the whole structure of indecomposable integers in non-monogenic simplest cubic fields possessing the integral basis (1) with $p = 3$ and $(k, l) = (1, 1)$. Note that this concrete subfamily already appears in the work of Cánovas Orvay [2]. The reason why we restrict to this subfamily is that the situation seems to be significantly different for each prime p. Moreover, this is the first time when the structure of indecomposable integers was derived for a family of non-monogenic number fields.

In the end, we will also mention several applications of this result. First, we will bound the number of variables of universal quadratic forms over these fields. Then, we will discuss the Pythagoras number of $\mathcal{O}_K$, which is equal to six as in the case when $\mathcal{O}_K = \mathbb{Z}[\rho]$. Furthermore, we will show that except for a few cases, the minimal norm of algebraic

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integers not associated with rational integers is still $2a+3$ as when $\mathcal{O}_K = \mathbb{Z}[\rho]$, which is the case which was discussed earlier by Lemmermeyer and Pethö [9].

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