FRACTAL TILES INDUCED BY TENT MAPS

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The presentation is based on joint work with K. Scheicher and V. Sirvent [5]. Consider a real number $\alpha > 1$, let $\beta = \beta(\alpha) := \alpha(\alpha-1)^{-1}$ and define the tent map T_α by

$$T_\alpha : [0,1] \longrightarrow [0,1], x \longmapsto \begin{cases} \alpha x & \text{if } x \in [0, \alpha^{-1}] \\ \beta(1-x) & \text{if } x \in [\alpha^{-1}, 1]. \end{cases}$$

The dynamics induced by tent maps has been studied intensively in [2, 3] and more recently in [4].

In the talk we present geometrical objects induced by the tent map that we call *tent tiles*. For this reason we suppose that α is a Pisot unit such that $\beta(\alpha)$ is also a Pisot unit. Let $d+1$ denote the algebraic degree of α, $A_\alpha \in \mathbb{R}^{d\times d}$ be a matrix similar to $\operatorname{diag}(\lambda_1, \ldots, \lambda_d)$, where $\lambda_1, \ldots, \lambda_d$ are the Galois conjugates different from α, and

$$\Psi_\alpha : \mathbb{Q}(\alpha) \longrightarrow \mathbb{R}^d$$

be the (uniquely determined) embedding that satisfies $\Psi_\alpha(\alpha x) = A_\alpha \Psi_\alpha(x)$ for all $x \in \mathbb{Q}(\alpha)$. Let $B_\alpha := A_\alpha(A_\alpha - I_d)^{-1} \in \mathbb{R}^{d\times d}$, where I_d denotes the $d \times d$ identity matrix. Then A_α and B_α are contractive matrices and the two functions

$$f_L : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \mathbf{x} \longmapsto A_\alpha \mathbf{x},$$
$$f_R : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \mathbf{x} \longmapsto B_\alpha(\Psi_\alpha(1) - \mathbf{x})$$

induce an iterated function system (IFS) in the sense of [1] in the Euclidean space $\mathbb{R}^d$. The tent tile $\mathcal{F}_\alpha$ is the invariant set induced by this IFS, that is the uniquely determined compact set that satisfies $\mathcal{F}_\alpha = f_L(\mathcal{F}_\alpha) \cup f_R(\mathcal{F}_\alpha)$.

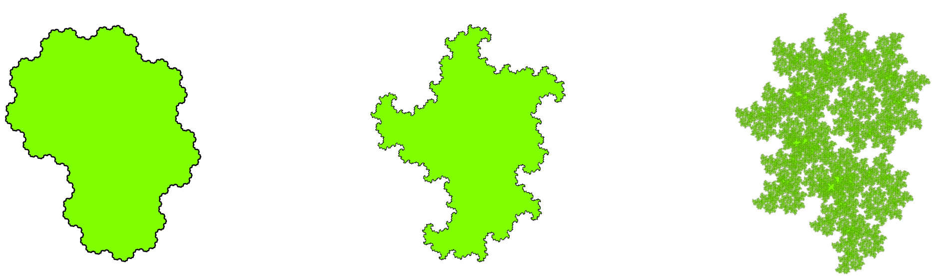

FIGURE 1. We see three of the six planar tent tiles.

A Pisot number α is called a *special Pisot number* when $\beta(\alpha)$ is also a Pisot number. Smyth showed in [6] that there exist exactly 11 special Pisot numbers where ten of them are algebraic units. Therefore, we only have ten tiles to study, two in $\mathbb{R}^1$, six in $\mathbb{R}^2$ (see Figure 1), and two in $\mathbb{R}^3$. In the talk we present results on the Hausdorff dimension of the boundary of these tent tiles, and we are concerned with periodic lattice tilings induced by them. It turns out that some tent tiles induce such a tiling, others do not. A third group induces a lattice tiling if we include the reflected tile (see Figure 2). Summing up, although their number is limited, tent tiles show up many different behaviours and properties. We also plan to outline the close relation of tent tiles with Rauzy fractals induced by substitutions and automorphisms of the free group. These connections are the basis for studying tent tiles.

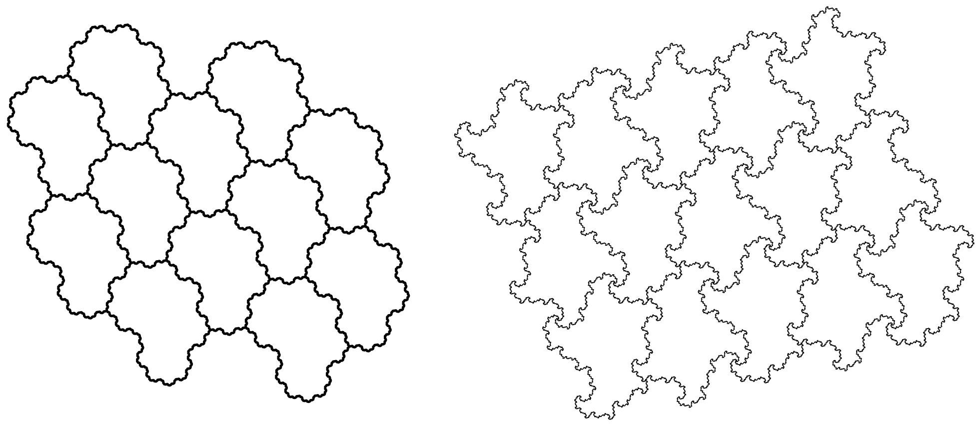

Figure 2. Two types of lattice tilings induced by tent tiles. On the left we see the tiling induced be the leftmost tile in Figure 1. The right tiling is induced by the tile depicted in the centre of Figure 1 and its reflection. The rightmost tile in Figure 1 does not induce a tiling.

References

[1] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., 30 (1981), pp. 713–747.

[2] J. C. Lagarias, H. A. Porta, and K. B. Stolarsky, *Asymmetric tent map expansions. I. Eventually periodic points*, J. London Math. Soc. (2), 47 (1993), pp. 542–556.

[3] ———, *Asymmetric tent map expansions. II. Purely periodic points*, Illinois J. Math., 38 (1994), pp. 574–588.

[4] K. Scheicher, V. F. Sirvent, and P. Surer, *Dynamical properties of the tent map.*, J. Lond. Math. Soc., II. Ser., 93 (2016), pp. 319–340.

[5] ———, *Fractal tiles induced by tent maps.*, Submitted, (2023).

[6] C. J. Smyth, *There are only eleven special Pisot numbers*, Bull. London Math. Soc., 31 (1999), pp. 1–5.

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