Almost sure limit theorems with applications to non-regular continued fraction algorithms

If we look at the interval $[0,1]$ and its representation as a continued fraction given by

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}},$$

then the digits (a_n) can be generated by a dynamical system by setting $a_n = \varphi \circ T^{n-1}$, where $\varphi(x) = \lfloor 1/x \rfloor$ and $T(x) = \{1/x\} = 1/x - \lfloor 1/x \rfloor$. T preserves a probability measure $\mathfrak{m}$ which is absolutely continuous with respect to Lebesgue and $\varphi \notin \mathcal{L}^1(\mathfrak{m})$. By this observation, we obtain, by a theorem by Aaronson, [Aar77] that there exists no norming sequence (γ_n) such that $\lim_{n\to\infty} \sum_{k=1}^n a_k(x)/\gamma_n = 1$ for almost every (a.e.) $x \in [0,1]$.

However, by a theorem by Diamond and Vaaler, [DV86], we obtain for a.e. $x \in [0,1]$ that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k(x) - \max_{1\le \ell \le n} a_\ell(x)}{n \log n} = \frac{1}{\log 2}. \tag{0.1}$$

Instead of the classical continued fraction expansion we aim to study the backward or Rényi type continued fraction expansion, see [Rén57]: Each $x \in [0,1]$ can be written as

$$x = 1 - \cfrac{1}{d_1(x) - \cfrac{1}{d_2(x) - \cfrac{1}{d_3(x) - \ddots}}}.$$

In a similar fashion, the entries (d_n) can be represented by a dynamical system, namely by setting $d_n(x) = (\chi \circ T_{BCF}^{n-1})(x)$, where $\chi(x) = \lfloor 1/(1-x) \rfloor + 1$ and $T(x) = \{1/(1-x)\}$.

Though the maps look very similar, from the metrical point of view they behave very differently. In particular, while the Gauss map T has a finite invariant measure $\mathfrak{m}$, the invariant measure μ which is invariant with respect to the T_{BCF} and absolutely continuous with respect to Lebesgue is infinite and σ-finite.

Hence, we are in the situation to study a system where $\mu([0,1]) = \infty$, $\int \chi \, \mathrm{d}\mu = \infty$ and, more specifically, we may set $E = [1/2, 1]$ and find that $\mu(E) < \infty$ and $\int_E \chi \, \mathrm{d}\mu = \infty$. Though there are strong laws of large numbers for non-integrable observables on an infinite measure space, see [LM18],[BS21], the particular situation of an observable not being integrable on a finite measure space has to the authors' knowledge not been studied before.

We obtain an almost sure limit result for a truncated sum of the backward continued fraction entries where we delete the maximal entry as in (0.1) and in some situations add additonal summands.

Similar results also hold for the even integer continued fractions and we give results to a general ergodic, conservative, measure presearving systems $(X, \mathcal{B}, m, \tau)$ and $\phi : X \to \mathbb{R}_{\ge 0}$, where $m(X) = \infty$ and there exists $E \in \mathcal{B}$ such that $\mu(E) < \infty$ and $\int_E \phi \, \mathrm{d}m = \infty$.

This is joint work in progress with Claudio Bonanno. The preprint will be available by the time of the conference.

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