On the binary digits of n and n^2

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Abstract

Let $s(n)$ denote the sum of digits in the binary expansion of the integer n. Hare, Laishram, and Stoll [4] studied the number of odd integers such that $s(n) = s(n^2) = k$ for a given positive integer k. These authors could not treat the remaining cases $k \in \{9, 10, 11, 14, 15\}$. In this talk, I will present the results of our article [1] on the cases $k \in \{9, 10, 11\}$ and the difficulties in settling for the two remaining cases $k \in \{14, 15\}$. Our proof is based on an efficient algorithm and new combinatorial techniques.

A related problem is to study perfect squares of odd integers with few binary digits. Szalay [5] showed that the set of solutions for three binary digits comprises a finite set $\{7, 23\}$ and one infinite family. The set of solutions for four binary digits is radically different since Corvaja and Zannier [3] and Bennett, Bugeaud, and Mignotte [2] independently proved that there are only finitely many solutions. The last authors conjectured that the set of solutions is $\{13, 15, 47, 111\}$. In the same paper [1], we have developed an algorithm that finds all solutions with a fixed sum of digits value. This algorithm supports the conjecture of [2] and shows new related results for perfect squares of odd integers with five binary digits.

This is joint work with K. Aloui, D. Jamet, H. Kaneko, S. Kopecki and T. Stoll.

References

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