THE DESCRIPTIVE COMPLEXITY OF THE SET OF POISSON GENERIC NUMBERS

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Years ago Zeev Rudnick introduced a property of real numbers called Poisson genericity: a real number is Poisson generic in an integer base b, if in its initial segments of its fractional expansion in bases b, the number of long words follows the Poisson distribution with parameter λ, for every positive real lambda. Peres and Weiss [8, 1] proved that almost all real mumbers, with respect to the Lebesgue measure, are Poisson generic.

For any integer base b greater than or equal to 2, we identify real number with their fractional expansion in base b.

Definition 1. An $x \in b^\omega$ is λ-Poisson generic if for every non-negative integer j we have $\lim_{k\to\infty} Z^\lambda_{j,k}(x) = e^{-\lambda}\frac{\lambda^j}{j!}$ where

$$Z^\lambda_{j,k}(x) = \frac{1}{b^k}\#\{w \in b^k\colon w \text{ occurs } j \text{ times in } x \restriction \lambda b^k + k\}.$$

A real number x is Poisson generic in base b if it is λ-Poisson generic in base b for every positive real λ.

Definition 2. A real number x is Borel normal in base b if for every block w of digits in base b,

$$\lim_{n\to\infty} \frac{\text{the number of occurrences of } w \text{ in } x \restriction n}{n} = b^{-|w|}.$$

It is known that $x \in b^\omega$ being Poisson generic implies that x is Borel normal in base b [8, 2]. Let $\mathcal{P}(b)$ be the set of Poisson generic numbers and let $\mathcal{N}(b)$ be the set of Borel normal numbers in base b.

In any topological space X, the collection of Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\mathbf{\Sigma}^0_1 =$ the open sets, $\mathbf{\Pi}^0_1 = \neg\mathbf{\Sigma}^0_1 = \{X - A\colon A \in \mathbf{\Sigma}^0_1\} =$ the closed sets, and for $\alpha < \omega_1$ we let $\mathbf{\Sigma}^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \mathbf{\Pi}^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\mathbf{\Pi}^0_\alpha = \neg\mathbf{\Sigma}^0_\alpha$. Alternatively, $A \in \mathbf{\Pi}^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha$, in particular $\mathbf{\Delta}^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. All of the collections $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions. A basic fact (see [6]) is that for any uncountable Polish space X, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarch can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) *hard* if $A \notin \mathbf{\Pi}^0_\alpha$ (resp. $A \notin \mathbf{\Sigma}^0_\alpha$). This says A is "no simpler" than a $\mathbf{\Sigma}^0_\alpha$ set. We say A is $\mathbf{\Sigma}^0_\alpha$-*complete* if $A \in \mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha$, that is, $A \in \mathbf{\Sigma}^0_\alpha$ and A is $\mathbf{\Sigma}^0_\alpha$ hard. This says A is exactly at the complexity level $\mathbf{\Sigma}^0_\alpha$. Likewise, A is $\mathbf{\Pi}^0_\alpha$-complete if $A \in \mathbf{\Pi}^0_\alpha - \mathbf{\Sigma}^0_\alpha$.

A fundamental result of Suslin (see [6]) says that in any Polish space $\mathcal{B}(X) = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1 = \neg \mathbf{\Sigma}^1_1$, and $\mathbf{\Sigma}^1_1$ is the pointclass of continuous images of Borel sets. Equivalently, $A \in \mathbf{\Sigma}^1_1$ iff A can be written as $x \in a \leftrightarrow \exists y\ (x, y) \in B$ where $B \subseteq X \times Y$ is Borel (for some Polish space Y). Similarly, $A \in \mathbf{\Pi}^1_1$ iff it is of the form $x \in A \leftrightarrow \forall y\ (x, y) \in B$ for a Borel B. The $\mathbf{\Sigma}^1_1$ sets are also called the *analytic* sets, and $\mathbf{\Pi}^1_1$ the *co-analytic sets*. We also have $\mathbf{\Sigma}^1_1 \neq \mathbf{\Pi}^1_1$ for any uncountable Polish space.

A set $D \subseteq X$ is in the class $D_2(\mathbf{\Pi}^0_3)$ if $D = A \setminus B$ where $A, B \in \mathbf{\Pi}^0_3$. A set D is $D_2(\mathbf{\Pi}^0_3)$-hard if $X \setminus D \notin D_2(\mathbf{\Pi}^0_3)$, and D is $D_2(\mathbf{\Pi}^0_3)$-complete if it is in $D_2(\mathbf{\Pi}^0_3)$ and is $D_2(\mathbf{\Pi}^0_3)$-hard. As with the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, the class $D_2(\mathbf{\Pi}^0_3)$ has a universal set and so is non-selfdual, that is, it is not closed under complements

H. Ki and T. Linton [7] proved that the set $\mathcal{N}(b)$ is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Further work was done by V. Becher, P. A. Heiber, and T. A. Slaman [3] who settled a conjecture of A. S. Kechris by showing that the set of absolutely normal numbers is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Furthermore, V. Becher and T. A. Slaman [4] proved that the set of numbers normal in at least one base is $\mathbf{\Sigma}^0_4(\mathbb{R})$-complete.

K. Beros considered sets involving normal numbers in the difference heirarchy in [5]. Let $\mathcal{N}_k(b)$ be the set of numbers normal of order k in base b. He proved that for $b \geq 2$ and $s > r \geq 1$, the set $\mathcal{N}_r(b) \backslash \mathcal{N}_s(b)$ is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$-complete (see [6] for a definition of the difference hierarchy). Additionally, the set $\bigcup_k \mathcal{N}_{2k+1}(2) \backslash \mathcal{N}_{2k+2}(2)$ is shown to be $\mathcal{D}_\omega(\mathbf{\Pi}^0_3)$-complete.

Our goal is to show the following result concerning the descriptive complexity of $\mathcal{P}(b)$.

Theorem 3. *$\mathcal{P}(b)$ is $\mathbf{\Pi}^0_3$-complete.*

We show that the notions of Poisson genericity and normality are "independent" by considering the complexity of the difference set.

Theorem 4. *$\mathcal{N}(b) \setminus \mathcal{P}(b)$ is D_2-$\mathbf{\Pi}^0_3$-complete.*

While $\mathcal{N}(b) \setminus \mathcal{P}(b)$ is D_2-$\mathbf{\Pi}^0_3$-complete, it is still unknown what its Hausdorff dimension is.

References

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