q-Recursive Sequences

Clemens Heuberger

Let $q \geq 2$ be an integer. We consider sequences where every subsequence whose indices run through a residue class modulo q^M is a linear combination of subsequences where for each of these subsequences, the indices run through a residue class modulo q^m for some $m < M$; i.e., a sequence x such that there are constants $M > m \geq 0$, $\ell \leq u$ and constants $c_{s,k}$ for all $0 \leq s < q^M$ and $\ell \leq k \leq u$ such that

$$x(q^M n + s) = \sum_{\ell \leq k \leq u} c_{s,k} x(q^m n + k)$$

holds for almost all non-negative integers n and all $0 \leq s < q^M$. We call such sequences q-recursive.

As an example, consider the number $p(n)$ of odd entries in the first n rows of Pascal's triangle. It can be shown that $p(2n) = 3p(n)$ and $p(2n+1) = p(n+1) + 2p(n)$ holds for all $n \geq 0$. In this example, we have $q = 2$, $M = 1$, $m = 0$, $\ell = 0$ and $u = 1$.

One of our main results is that every q-recursive sequence is in fact q-regular in the sense of Allouche and Shallit, see also the keynote by Daniel Krenn at this conference. These are sequences which can be written as the first component of a vector valued sequence v such that there are square matrices A_s for $0 \leq s < q$ such that $v(qn + s) = A_s v(n)$ holds for all non-negative integers n and for all $0 \leq s < q$. The asymptotic behaviour of such sequences has been thoroughly studied.

The result remains true if inhomogeneities are included into the recurrence, i.e. if there are q-regular sequences g_s for $0 \leq s < q$ such that

$$x(q^M n + s) = \sum_{\ell \leq k \leq u} c_{s,k} x(q^m n + k) + g_s(n).$$

For instance, the binary sum of digits function s satisfies a recurrence of this type: we have $s(2n) = s(n)$ and $s(2n+1) = s(n) + 1$ for all $n \geq 0$. Another example is the number $M(n)$ of comparisons when sorting n numbers with merge sort: there, we have $M(2n) = 2M(n) + 2n - 1$ and $M(2n+1) = M(n) + M(n+1) + 2n$ for all $n \geq 1$.

The result is of algorithmic nature and the "linear representation" of the regular sequence is given explicitly. Furthermore, the result is also implemented in SageMath.

The linear representations tend to be of rather high dimension; therefore, a minimisation of the dimension is desirable. We discuss the relevance of the minimisation algorithm by Berstel and Reutenauer for recognisable series for q-regular sequences; it is also implemented in SageMath.

In this talk, I also intend to give several examples of q-recursive sequence and their asymptotic behaviour.

(Based on joint work with D. Krenn and G. Lipnik)