COUNTING RATIONALS AND DIOPHANTINE APPROXIMATION ON MISSING-DIGIT SETS

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1. A fractal analogue of Serre's dimension growth conjecture

Recall Serre's dimension growth conjecture, which has since been proved by Salberger, see [3].

Theorem 1.1 (Salberger, 2009). *Let $m, d \geq 2$, and let $X \subseteq \mathbb{P}^m$ be an integral projective variety of degree d over $\mathbb{Q}$. Then, for $T \geq 1$ and $\varepsilon > 0$, the number of rational points on X up to height T is*

$$O_{X,\varepsilon}(T^{\dim(X)+\varepsilon}).$$

This problem has strongly influenced the theory of rational points, in particular the development of the determinant method, and work on uniform aspects continues.

We consider a fractal analogue proposed in [1]. For compact $K \subset \mathbb{R}$ and for $T \geq 1$, define

$$\mathcal{N}_K(T) = \#\{x \in K : qx \in \mathbb{Z} \text{ for some } q \in \mathbb{Z} \cap [1, T]\}.$$

Conjecture 1.2 (Broderick–Fishman–Reich, 2011). *Let K be the middle-third Cantor set and $\varepsilon > 0$. Then*

$$\mathcal{N}_K(T) \ll_\varepsilon T^{\kappa+\varepsilon},$$

where $\kappa = \frac{\log 2}{\log 3}$ is the Hausdorff dimension of K.

Heuristic and empirical data supporting this can be found in [4]. The lower bound $\mathcal{N}_K(T) \gg T^\kappa \log T$ follows from constructing eventually periodic ternary expansions, whilst the upper bound $\mathcal{N}_K(T) \ll T^{2\kappa}$ follows from the separation of rationals and a regularity property of the Cantor measure. Despite the problem having received significant attention from leading researchers, nothing further is known about this counting function.

For integers $b \geq 3$ and $a \in [0, b-1]$, denote by $K_{b,a}$ the set of $x \in [0,1]$ that can be written without the digit a in base b, and note that $K_{3,1}$ is the middle-third Cantor set. We conjecture that if $K = K_{b,a}$ then $\mathcal{N}_K(T) \ll_\varepsilon T^{\kappa+\varepsilon}$, where $\kappa = \frac{\log(b-1)}{\log b}$ is the Hausdorff dimension of K. The separation of rationals again yields $\mathcal{N}_K(T) \ll T^{2\kappa}$.

Theorem 1.3 (ACVY). *Let $K = K_{b,a}$ with $b \geq 5$ or $(b, a) \in \{(4, 0), (4, 3)\}$. Then there exists $\rho > 0$ such that $\mathcal{N}_K(T) \ll_\rho T^{2\kappa-\rho}$.*

2. Applications to diophantine approximation on fractals

2.1. Intrinsic diophantine approximation.

In the twilight of his career, Mahler proposed "some suggestions for further research", asking how closely the elements of $K_{3,1}$ can be approximated:

- by rationals;
- by rationals in $K_{3,1}$ (intrinsic approximation).

For compact $K \subset \mathbb{R}$ and for $\alpha > 1$, denote by $W_K(\alpha)$ the set of $x \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| < q^{-\alpha}, \qquad \frac{p}{q} \in K$$

holds for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$, where $\mathbb{N} = \mathbb{Z}_{>0}$. Put

$$\mathbf{VWA}_K = \bigcup_{\alpha > 1} W_K(\alpha).$$

To each $K_{b,a}$, we can associate a natural probability measure.

Conjecture 2.1 (Broderick–Fishman–Reich, 2011)**.** *The set* $\mathbf{VWA}_K$ *has measure zero.*

It follows from [5] that if $K = K_{3,1}$ and $\alpha > 2$ then $W_K(\alpha)$ has measure zero.

Theorem 2.2 (ACVY)**.** *Let* $K = K_{b,a}$ *with* $b \geq 5$ *or* $(b, a) \in \{(4, 0), (4, 3)\}$*. Then there exists* $\rho > 0$ *such that* $W_K(2 - \rho)$ *has measure zero.*

2.2. **The distribution of the irrationality exponent.** The *irrationality exponent* of $x \in \mathbb{R}$, denoted $w(x)$, is the supremum of $w \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| < q^{-w}$$

holds for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Define

$$\mathcal{M}(w) = \{x \in \mathbb{R} : w(x) \geq w\}$$

and

$$\mathbf{VWA} = \{x \in \mathbb{R} : w(x) > 2\} = \bigcup_{\alpha > 2} \mathcal{M}(w).$$

The following conjecture was proposed by Bugeaud and Durand in 2016 in the case of the middle-third Cantor set, generalising a conjecture of Levesley, Salp and Velani, see [2]. We write $\dim_{\mathrm{H}}$ for the Hausdorff dimension.

Conjecture 2.3 (Levesley–Salp–Velani, 2007, and Bugeaud–Durand, 2016)**.** *Let* K *be a missing-digit set of Hausdorff dimension* κ*. Then*

$$\dim_{\mathrm{H}}(\mathcal{M}(w) \cap K) = \max\left\{ \frac{2}{w} + \kappa - 1, \frac{\kappa}{w} \right\} \qquad (w \geq 2) \tag{1}$$

and

$$\dim_{\mathrm{H}}(\mathbf{VWA} \cap K) = \dim_{\mathrm{H}}(K). \tag{2}$$

In a notable breakthrough, Yu [6] established (2), as well as (1) for w in some interval containing 2, when $K_{b,a}$ and b is very large.

Theorem 2.4 (ACVY)**.** *Let* $K = K_{b,a}$ *with* $b \geq 7$*. Then* $\rho > 0$ *such that* (1) *holds for* $2 \leq w \leq 2+\rho$*. In particular, we have* (2)*.*

3. The Fourier ℓ^1 dimension

We use Fourier analysis. Let ν be a Borel probability measure on $[0,1]$, and write

$$\hat{\nu}(\xi) = \int_0^1 e(-\xi x)\mathrm{d}\nu(x),$$

where $e(y) = e^{2\pi i y}$. By extending periodically, we can consider Fourier series expansions. Fourier ℓ^t dimensions were introduced in [6] to describe averaged decay rates of the Fourier coefficients of a measure. For $t \geq 1$, the *Fourier ℓ^t dimension* of ν is

$$\kappa_t(\nu) = \sup\left\{ s : \sum_{\xi=0}^{Q} |\hat{\nu}(\xi)|^t \ll Q^{1-s} \right\}.$$

For missing-digit measures, we have

$$\frac{\kappa_2(\nu)}{2} \leq \kappa_1(\nu) \leq \kappa_2(\nu) = \dim_{\mathrm{H}}(\nu).$$

The quantity $\kappa_1(\nu)$ plays an essential role in our work, where ν is the natural probability measure associated to the missing-digit set. For general measures ν, the problem of estimating the Fourier ℓ^1 dimension is intractable. However, the classical product formula for $\hat{\nu}$ enables us to estimate $\kappa_1(\nu)$ to arbitrary precision via induction up the tower of rings $\mathbb{Z}/b^N\mathbb{Z}$.

References

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