Partitions into powers of an algebraic number

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Integer partitions

A **partition** of $n \in \mathbb{Z}_{\geq 0}$ is a way of writing n as a sum

$$n = a_1 + a_2 + \cdots + a_j, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$

Two sums that differ only in the order of parts are considered the same.
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Partition identities, congruence properties, parity, . . .

Let $m \in \mathbb{Z}$, $m \geq 2$. An m-**ary partition** of n is an expression of the form

$$n = a_j m^j + a_{j-1} m^{j-1} + \cdots + a_1 m + a_0, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$

Let $b_m(n)$ denote the m-**ary partition function**.

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$$b_m(nm) = b_m(mn-1) + b_m(n)$$

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Congruence properties: Andrews, Fraenkel, and Sellers (2015)
If $n = a_j m^j + \cdots + a_1 m + a_0$ is the base m representation of n, so that $a_i \in \{0, 1, \ldots, m-1\}$, then

$$b_m(mn) \equiv \prod_{i=0}^{j} (a_i + 1) \pmod{m}.$$

Mahler's asymptotics was extended to non-integer β by de Bruijn (1948) and Pennington (1953).
Let $\beta \in \mathbb{R}$, $\beta > 1$. Define $P_\beta(x)$ as the number of solutions to

$$a_j\beta^j + a_{j-1}\beta^{j-1} + \cdots + a_1\beta + a_0 < x, \qquad j, a_i \in \mathbb{Z}_{\geq 0}.$$

Then

$$\log(P_\beta(x) - P_\beta(x-1)) \sim \log P_\beta(x) \sim \frac{(\log x)^2}{2\log\beta}.$$

Partitions into powers of $\beta \in \mathbb{R}$

Definition

Let $\beta \in \mathbb{R} \setminus \{-1, 0, 1\}$. A partition of $\alpha \in \mathbb{R}$ into powers of β is an expression of the form

$$\alpha = a_j \beta^j + a_{j-1} \beta^{j-1} + \cdots + a_1 \beta + a_0, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$

Let $p_\beta(\alpha)$ be the number of partitions of α.

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- The number of partitions of α can be infinite (e.g. for $\beta = 1/2$ or $\beta = -2$).
- It can also be zero.

Question 1: For which β does p_β attain only finite values?
Question 2: What can we say about the range of p_β?

Finiteness of the partition function

Question 1: For which β does p_β attain only finite values?

Sufficient condition: $\beta > 1$. This condition is not necessary.
Observe that if β is transcendental, then $p_\beta(\alpha) = 0$ or 1 for every $\alpha \in \mathbb{R}$.
We focus on the case when $\beta \in \mathbb{R}$ is a root of a quadratic polynomial $Ax^2 + Bx + C$ over $\mathbb{Z}$, and we assume

- $A > 0$
- the polynomial is irreducible in $\mathbb{Z}[x]$
- in particular, $\gcd(A, B, C) = 1$
- $\Delta = B^2 - 4AC > 0$

Finiteness of the partition function

Observe: If one of the conjugates β, β' is greater than 1, then $p_\beta(\alpha) < \infty$ for every $\alpha \in \mathbb{R}$.
If

$$\alpha = a_j\beta^j + a_{j-1}\beta^{j-1} + \cdots + a_1\beta + a_0,$$

let

$$\alpha' = a_j(\beta')^j + a_{j-1}(\beta')^{j-1} + \cdots + a_1(\beta') + a_0.$$

Partitions of α into powers of β are in one-to-one correspondence with partitions of α' into powers of β', hence $p_\beta(\alpha) = p_{\beta'}(\alpha')$.

Finiteness of the partition function

Theorem 1

Let $\beta \in \mathbb{R}$ be a root of a quadratic polynomial $Ax^2 + Bx + C$ with $A > 0$ which is irreducible in $\mathbb{Z}[x]$. Then the following are equivalent.

1. For every $\alpha \in \mathbb{R}$: $p_\beta(\alpha) < \infty$,
2. either $2A + B \leq 0$, or $[2A + B > 0$ and $A + B + C < 0]$,
3. at least one of the conjugates β and β' is greater than 1.

Proof:
(ii) $\Leftrightarrow$ (iii): easy
(iii) $\Rightarrow$ (i): done
It remains to prove (i) $\Rightarrow$ (ii).

Proof of Theorem 1

We are proving

$$[2A + B > 0 \text{ but } A + B + C \geq 0] \Rightarrow \exists \alpha \in \mathbb{R} : p_\beta(\alpha) = \infty$$

Proof by case distinction.

Case I. $B \geq 0$ and $C > 0$. We get

$$0 = A\beta^2 + B\beta + C,$$

which is a non-trivial partition of 0. We also get

$$-1 = A\beta^2 + B\beta + (C - 1).$$

Conclusion: $p_\beta(\alpha) > 0$ if and only if $\alpha \in \mathbb{Z}[\beta]$. For every $\alpha \in \mathbb{Z}[\beta]$, $p_\beta(\alpha) = \infty$.

Proof of Theorem 1

We are proving

$$[2A + B > 0 \text{ but } A + B + C \geq 0] \Rightarrow \exists \alpha \in \mathbb{R} : p_\beta(\alpha) = \infty$$

Case II. $B \geq 0$ and $C < 0$. Let $C = -F$, F positive, so that β is a root of $Ax^2 + Bx - F$. We get

$$F = A\beta^2 + B\beta$$

and $A + B \geq F$. There are infinitely many partitions of $A\beta + F$:

$$\begin{aligned} A\beta + F &= A\beta^2 + (A + B)\beta \\ &= A\beta^2 + (A + B - F)\beta + F\beta = A\beta^3 + (A + B)\beta^2 + (A + B - F)\beta \\ &= \dots \end{aligned}$$

Case III. $B < 0$ and $C < 0$. This is similar to II.

Proof of Theorem 1

We are proving

$$[2A + B > 0 \text{ but } A + B + C \geq 0] \Rightarrow \exists \alpha \in \mathbb{R} : p_\beta(\alpha) = \infty$$

Case IV: $B < 0$ and $C > 0$. This is the case when $\beta > 0$ and $\beta' > 0$. If we let $B = -E$, E positive, then

$$E\beta = A\beta^2 + C,$$

and we assume $A + C > E$. It is not clear what to rewrite. We need a new idea for counting partitions.

Idea for counting partitions

Suppose that $\alpha \in \mathbb{R}$ is expressed as

$$\alpha = c_j\beta^j + c_{j-1}\beta^{j-1} + \cdots + c_1\beta + c_0, \qquad c_i \in \mathbb{Z}_{\geq 0}.$$

If α has another partition

$$\alpha = b_k\beta^k + b_{k-1}\beta^{k-1} + \cdots + b_1\beta + b_0, \qquad b_i \in \mathbb{Z}_{\geq 0},$$

then we let Q, R denote the two polynomials

$$Q(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0,$$
$$R(x) = c_j x^j + c_{j-1}x^{j-1} + \cdots + c_1 x + c_0.$$

It follows that the minimal polynomial of β divides $Q(x) - R(x)$. Thus there exists a polynomial P such that

$$P(x)(Ax^2 + Bx + C) = Q(x) - R(x).$$

The coefficient of x^i in $Q(x) - R(x)$ is $\geq -c_i$. Conversely, if we find a polynomial P such that the coefficients of $P(x)(Ax^2 + Bx + C)$ satisfy this bound, we can reconstruct a partition of α.

Finishing the proof of Theorem 1

Back to case IV. Recall that β is a root of $Ax^2 + Bx + C$. We assume

$$A > 0, \quad B < 0, \quad C > 0, \quad 2A + B > 0, \quad A + B + C > 0$$

and show: There exists $c \in \mathbb{Z}_{\geq 1}$ such that $p_\beta(c\beta) = \infty$.
We use the preceding idea for counting partitions and prove that there exist infinitely many polynomials $P(x) \in \mathbb{Z}[x]$ such that

$$P(x)(Ax^2 + Bx + C) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

satisfies

- $b_n > 0$,
- $b_i \geq 0$ for $i \neq 1$,
- $b_1 = -c$.

□

Question 2: What can we say about the range of the partition function?

Terminology.

- β is called a quadratic integer if it is a root of a monic irreducible polynomial $x^2 + Bx + C$ over $\mathbb{Z}$
- $\operatorname{Tr} \beta = \beta + \beta' = -B, \quad \operatorname{N} \beta = \beta\beta' = C$
- if $\beta > 0$ and $\beta' > 0$, then we say that β is totally positive

Range of the partition function

Theorem 2

If a totally positive quadratic integer β satisfies

$$\operatorname{Tr}\beta \leq \operatorname{N}\beta < 2\operatorname{Tr}\beta - 4,$$

then for every $n \in \mathbb{Z}_{\geq 0}$

$$p_\beta((\operatorname{Tr}\beta)\beta^n) = n + 1.$$

Corollary

In a real quadratic field $K = \mathbb{Q}(\sqrt{D})$, there exist infinitely many β such that p_β attains all non-negative integer values.

Range of the partition function

Theorem 2

If a totally positive quadratic integer β satisfies

$$\operatorname{Tr}\beta \leq \operatorname{N}\beta < 2\operatorname{Tr}\beta - 4, \tag{1}$$

then for every $n \in \mathbb{Z}_{\geq 0}$

$$p_\beta((\operatorname{Tr}\beta)\beta^n) = n + 1. \tag{2}$$

Remarks.

- It is not difficult to show that $(\operatorname{Tr}\beta)\beta^n$ has **at least** $n+1$ partitions.
- The hard part is to show that there are no other partitions. For this, we use the idea for counting partitions from before.
- The bounds in Theorem 2 are optimal in the sense that if one of the inequalities in (1) does not hold, then the conclusion (2) does not hold.

Open questions

1. Theorem 1 shows that if $\beta \in \mathbb{R}$ is quadratic, then

$$[\forall \alpha \in \mathbb{R} : p_\beta(\alpha) < \infty] \Leftrightarrow [\beta > 1 \text{ or } \beta' > 1].$$

   Is it possible to generalize this to higher degrees?

2. Let β be a totally positive quadratic integer. Is it true that for every $n \in \mathbb{Z}_{\geq 0}$, there exists $\alpha \in \mathbb{R}$ such that $p_\beta(\alpha) = n$? It would be interesting to know the answer even in specific examples, e.g. $\beta = 2 + \sqrt{2}$.

3. Does there exist a quadratic integer β with this property which is not totally positive?

Thank you for your attention!