Varying the base of Numeration in the Rényi Numeration Dynamical System

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Rényi ('57) : introduced the numeration dynamical system

$$([0,1], T_\beta)$$

with the β-transformation

$$T_\beta : x \to \{\beta x\}$$

for any $\beta > 1$.

The study *"β tending to 1^+, the base of numeration β running over the set of reciprocal algebraic integers (if any)"*, entirely contains the

Problem of Lehmer (stated in 1933)

and allows to solve the Conjecture of Lehmer.

Talk :

- Varying the base of numeration, $1 < \beta < (1+\sqrt{5})/2$, β tending to 1, $\zeta_{\beta}(z)$,

- Mahler measure

- Main results and ex-Lehmer's Conjecture

- Rewriting trails and Kala-Vavra's Theorem

- Localization of the conjugates invovlved in the minorant of the Mahler measure, i.e. the lenticular zeroes : an analytic curve with infinitely many branches

Idea : 2009, first version 2017, last version : 2021.

Dynamical zeta function of T_β

Artin-Mazur (1965) :

Theorem : Let $f : M \to M$ a diffeomorphism of a compact variety M, such that the iterates f^k have fixed points which are *isolated*. Denote

$$\zeta_f(z) := \exp\left(\sum_{k=1}^{\infty} \frac{\#\ \mathrm{Fix}\ f^k}{k} z^k\right).$$

For a dense subset of $f \in$Diff(M), this expression defines an analytic function in a neighbourhood of 0.

Fredholm (1903) and Grothendieck (1956).
Following A. Weil's $\zeta_{Frobenius}(z)$, and Weil's Conjectures.
Smale, Manning, Ruelle, Fried, ... : rational functions (under some assumptions).

Case of the Rényi numeration system : related to $d_\beta(1) = 1 + \sum_{i\geq 1} t_i \beta^{-i}$ by the denominators :

Theorem(Takahashi, Ito ; Flatto, Lagarias, Poonen)[Ergodic theory]
(i) The zeta function $\zeta_\beta(z)$ of the β-transformation is given by

$$\zeta_\beta(z) = \frac{1 - z^N}{(1-\beta z)(\sum_{n=0}^{\infty} T_\beta^n(1) z^n)}$$

where N is the minimal value such that $T_\beta^N(1) = 0$, and if no such iterate exists, then $z^N \cong 0$,

(ii) $\zeta_\beta(z)$ is a meromorphic function in the open unit disc. It is holomorphic in $|z| < 1/\beta$, has a simple pole at $z = 1/\beta$,

(iii) if the sequence of values $\{T_\beta^n(1) \mid n = 1, 2, \ldots\}$ is eventually periodic, then $\zeta_\beta(z)$ is a rational function and continues meromorphically to $\mathbb{C}$. Otherwise, it has the unit circle $|z| = 1$ as natural boundary to analytic continuation.

Varying the base of numeration : in $1 < \beta < (1+\sqrt{5})/2$

$$-(1-\beta z)\left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n\right) = -1 + t_1 z + t_2 z^2 + t_3 z^3 + \ldots$$

with the digits coming from the orbit of 1 under T_β :

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\}\rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\}\rfloor, \ldots, t_i = \lfloor \beta T_\beta^{i-1}(1)\rfloor, i \geq 1.$$

- alphabet $= \{0, 1\}$,
- complete lexicographical ordering (Parry '60)

$$1 < \gamma < \beta \qquad \text{iff} \qquad (t'_1, t'_2, t'_3, \ldots) <_{lex} (t_1, t_2, t_3, \ldots)$$

where the Rényi γ-expansion of 1 is $d_\gamma(1) = 0.t'_1 t'_2 t'_3 \ldots$,
resp. the Rényi β-expansion of 1 is $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$.

- the analytic function $\zeta_\beta(z)$ is defined at least on $D(0,1)$ (radius of cvgce is 1 in Artin-Mazur's thm), obeys the Carlson-Polya dichotomy :

• either a rational function, or

• the unit circle is a natural boundary,

- self-admissibility (Conditions of Parry) implies lacunarity a minima controlled by n only :

any $\beta \in (1, (1+\sqrt{5})/2)$ corresponds uniquely to an element of :

$$\{-1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} + \ldots :$$

$$n \geq 3, m_1 - n \geq n-1, m_q - m_{q-1} \geq n-1 \quad \text{for} \quad 2 \leq q\}.$$

(named Parry Upper functions at β)

Because the set of simple Parry numbers is dense in $(1,+\infty)$, it is sufficient to consider the class of lacunary almost-Newman integer polynomials

$$\mathscr{C} := \{-1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} :$$
$$n \geq 3,\, m_1 - n \geq n-1, m_q - m_{q-1} \geq n-1 \quad \text{for} \quad 2 \leq q \leq s\}.$$

equipped with the total (lexicographical) ordering,
and allow the completion.

Contains all the trinomials $-1 + x + x^n, n \geq 3$.

Varying the base of numeration $\beta \longleftrightarrow$ varying lexicographically in $\mathscr{C}$.

let θ_n be the unique root of the trinomial $G_n(z) := -1 + z + z^n$ in $(0,1)$. We have : $\lim_n \theta_n^{-1} = 1$.

β reciprocal in between :

$- -$

$$\theta_n^{-1} \qquad\qquad -1 + z + z^n$$

$$\theta_n^{-1} < \beta < \theta_{n-1}^{-1} \qquad -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} + \ldots$$

where $m_1 - n \geq n - 1$,

$m_{q+1} - m_q \geq n - 1$

$$\theta_{n-1}^{-1} \qquad\qquad -1 + z + z^{n-1}$$

$- -$

The relation $<$ is transformed into a lexicographical perturbation of the trinomials.

$$\beta \to 1^+ \qquad \Longleftrightarrow \qquad n = \mathrm{dyg}(\beta) \to \infty.$$

Def. : n is called the **dynamical degree of** β, denoted by $\mathrm{dyg}(\beta)$.

Strategy of proof :

• solve Lehmer's Conjecture using the zeroes of the trinomials, of modulus < 1 (poles of $\zeta_{\theta_n^{-1}}(z)$) [VG '16],

• extend the method to all $P \in \mathscr{C}$, viewed as lexicographically perturbed trinomials, from a subcollection of zeroes, called lenticular (lenticular poles of $\zeta_\beta(z)$).

Needs : exact expressions (asymptotic expansions) of the roots, localization of zeroes, factorization, as functions

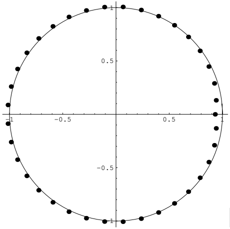
- of n (lacunarity) and $(m_1, \ldots, m_s)$,

- of the zeroes of $-1 + x + x^n$, when n and s become large, expressed as explicit functions of n, with :

$\rightarrow$ the study of the limit, n fixed, for s tending to ∞, and n tending to infinity.

Example with $n = 37$:

$$P(x) := (-1 + x + x^{37}) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649}$$

a) 

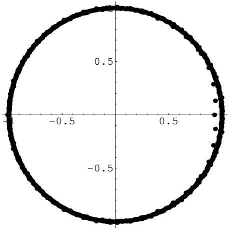
b) 

FIGURE: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $P(x) = G_{37}(x) + \ldots + x^{649}$. The lenticulus of roots of P is obtained by a very slight deformation of the restriction of the lenticulus of roots of G_{37} to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of P can be found in a narrow annular neighbourhood of $|z| = 1$.

Mahler measure

Definition : Mahler measure for

$$P(X) = a_0(X-\alpha_1)(X-\alpha_2)\ldots(X-\alpha_n) =$$
$$a_0X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n \in \mathbb{Z}[X], \qquad a_0a_n \neq 0$$

then

$$\mathrm{M}(P) := |a_0| \prod_{i,|\alpha_i|\geq 1} |\alpha_i|.$$

multiplicativity : $P = P_1 \times P_2 \times \ldots \times P_m, \Rightarrow \mathrm{M}(P) = \mathrm{M}(P_1)\ldots\mathrm{M}(P_m)$.

ex. : $P = \Phi_1 \times \ldots \times \Phi_r \times R$ with R irr. pol., Φ_j cyclot. $\Longrightarrow \mathrm{M}(P) = \mathrm{M}(R)$.

Def. : α alg. number, deg $\alpha = n$, P_α his minimal polynomial,

$$\mathrm{M}(\alpha) := \mathrm{M}(P_\alpha).$$

facts : $\mathrm{M}(\alpha) \geq 1$, $\mathrm{M}(\alpha) = \mathrm{M}(\alpha^{-1})$,

$\mathrm{M}(\alpha) = 1$ if α is a root of unity (+ Kronecker's Theorem, 1857).

Adler Marcus (1979) (topological entropy and equivalence of dynamical systems), Perron-Frobenius theory) :

$$\{\mathrm{M}(\alpha) \mid \alpha \text{ alg. number}\} \subset \mathbb{P}_{Perron},$$
$$\{\mathrm{M}(P) \mid P \in \mathbb{Z}[X]\} \subset \mathbb{P}_{Perron}.$$

Two strict inclusions (Dubickas 2004, Boyd 1981).

Definition : $\alpha \in \mathbb{P}_{Perron}$ if $\alpha = 1$ or if $\alpha > 1$ is a real algebraic integer, for which the conjugates $\alpha^{(i)}$ satisfy $|\alpha^{(i)}| < \alpha$ (i.e. dominant root > 1).

About the minorant ?

Lehmer's problem (1933)

in the exhaustive search for large prime numbers : *if ε is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$$

where the a_is are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 *and* $1+\varepsilon$*... Whether or not the problem has a solution for* $\varepsilon < 0.176$ *we do not know*.

Lehmer's strategy : P_α with small M : useful to obtain large prime numbers p, in the Pierce numbers of α. Iwasawa theory : large powers of primes. Einsiedler, Everest and Ward : study of the density of such ps.

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Conjecture of Lehmer : there exists $c > 0$ such that

$$\mathrm{M}(\alpha) \geq 1 + c$$

for any algebraic number $\alpha \neq 0$ which is not a root of unity,

$->$ values : discontinuity at 1 (meaning, sense, of c ?).

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Lehmer's Problem is a limit problem + restrictions :

$$\mathrm{M}(P) := |a_0| \prod_{i,|\alpha_i|\geq 1} |\alpha_i| \quad \Longrightarrow \quad \mathrm{M}(P) := |a_0| \geq |a_0|.$$

Let $\alpha \in \overline{\mathbb{Q}},\ P = P_\alpha$:

- $*$ if $\alpha \in \overline{\mathbb{Q}} \setminus \mathscr{O}_{\overline{\mathbb{Q}}}$, then $|a_0| \geq 2 \quad \Longrightarrow \quad \mathrm{M}(P) \geq 2$,
- $*$ if α is an algebraic integer which is not reciprocal ($P_\alpha \neq P_\alpha^*$ with

$$P_\alpha^*(X) = X^{\deg P_\alpha} P_\alpha(1/X) \quad),$$

Smyth's Theorem '71 $\Longrightarrow$ $\mathrm{M}(P_\alpha) \geq \Theta = 1.32\ldots$ (= smallest Pisot number, $X^3 - X - 1$ mini. pol.).

- $*$ <u>restriction to real reciprocal algebraic integers is sufficient</u> : if $\alpha \in \mathscr{O}_{\overline{\mathbb{Q}}}$, which is reciprocal ($P_\alpha = P_\alpha^*$), consider its house

$$\max\{|\alpha_i|\} =: \overline{|\alpha|} \in \mathscr{O}_{\overline{\mathbb{Q}}}$$

which is real, ≥ 1.

What is known ?   deg(α) tends to infinity if $\lceil\alpha\rceil > 1$ tends to 1, by :

Northcott's Theorem : for all $B \geq 0$, $d \geq 1$,

$$\#\{\alpha \in \overline{\mathbb{Q}} \mid \mathrm{Log}\,(\mathrm{M}(\alpha))/\deg(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < +\infty.$$

in Dio. Geom. : bound on "degree" + bound on "h" gives finiteness property (Mordell eff., etc).

Dobrowolski's inequality ('79) : for any reciprocal algebraic integer α of degree d,

$$\mathrm{M}(\alpha) > 1 + (1-\varepsilon)\left(\frac{LogLog\ d}{Log\ d}\right)^3, \qquad d > d_1(\varepsilon).$$

(Dobrowolski, $1/1200$, Schinzel, $1-\varepsilon$ for $d > d_1$).

Here, the lower bound in the rhs tends to 1 when d tends to infinity. Remarkable inequality. Not satisfying.

Assumption : existence of a real reciprocal algebraic integer $\beta > 1$ having : $\mathrm{M}(\beta) < 1.176280\ldots$ Lehmer's number.

$P_\beta(z)$ denom. of $\zeta_\beta(z)$

β

Attack : $\beta > 1$ real reciprocal algebraic integer,

$$1 < \beta \leq \boxed{\beta} \leq \mathrm{M}(\beta).$$

β tends to 1^+,
$\boxed{\beta}$ tends to 1^+.
? non-trivial minimum of $\beta \to \mathrm{M}(\beta)$, of $\boxed{\beta} \to \mathrm{M}(\boxed{\beta})$, when $\mathrm{dyg}(\beta) = n \to +\infty$.

+
identification of the lenticular poles of $\zeta_\beta(z)$ as lenticular conjugates of β.

Denote $G_n(x) := -1 + x + x^n, \ \ n \geq 3.$

Theorem

Let χ_3 be the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0, 1$ or $2 \pmod 3$), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol), and denote $L(s, \chi_3) = \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3. Then

$$\lim_{n \to +\infty} \mathrm{M}(G_n) = \exp\left(\frac{3\sqrt{3}}{4\pi}\mathrm{L}(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi}\int_0^{\pi/3} \mathrm{Log}\left(2\sin\left(\frac{x}{2}\right)\right) dx\right)$$

$$= 1.38135\ldots =: \Lambda. \tag{1}$$

Theorem

Let $n \geq 2$ be an integer. Then,

$$\mathrm{M}(-1+X+X^n) \;=\; \left(\lim_{m\to+\infty} \mathrm{M}(G_m)\right)\left(1+\frac{s(n)}{n^2}+O(n^{-3})\right) \qquad (2)$$

with, for n odd :

$$s(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\ldots & \text{if } n \equiv 1 \text{ or } 3 \pmod 6, \\ -\sqrt{3}\pi/6 = -0.9069\ldots & \text{if } n \equiv 5 \pmod 6, \end{cases}$$

for n even :

$$s(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\ldots & \text{if } n \equiv 0 \text{ or } 4 \pmod 6, \\ +\sqrt{3}\pi/12 = +0.4534\ldots & \text{if } n \equiv 2 \pmod 6. \end{cases}$$

Why mod 6 ?

Theorem (Selmer)

Let $n \geq 2$. If $n \not\equiv 5 \pmod 6$, then $G_n(X)$ is irreducible. If $n \equiv 5 \pmod 6$, then the polynomial $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor in its factorization and $G_n(X)/(X^2 - X + 1)$ is irreducible.

Factorization using Ljunggren's Lemma

Theorem (Dutykh-VG, '18)

For any $f \in \mathscr{C}$, $n \geq 3$, denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where

A is the cyclotomic component,

B the reciprocal noncyclotomic component,

C the nonreciprocal part.

Then C is irreducible.

(generalizes Selmer's Theorem). Heuristics (Monte-Carlo) :
A is found often trivial (75%), B is conjectured to be inexistant.

Proposition :
If $P(z) \in \mathbb{Z}[z]$, $P(1) \neq 0$, is nonreciprocal and irreducible, then $P(z)$ has no root of modulus 1.

Proof : Let $P(z) = a_d z^d + \ldots + a_1 z + a_0$, $a_0 a_d \neq 0$, be irreducible and nonreciprocal. We have $\gcd(a_0, \ldots, a_d) = 1$. If $P(\zeta) = 0$ for some ζ, $|\zeta| = 1$, then $P(\overline{\zeta}) = 0$. But $\overline{\zeta} = 1/\zeta$ and then $P(z)$ would vanish at $1/\zeta$. Hence P would be a multiple of the minimal polynomial P^* of $1/\zeta$. Since $\deg(P) = \deg(P^*)$ there exists $\lambda \neq 0, \lambda \in \mathbb{Q}$, such that $P = \lambda P^*$.
In particular, looking at the dominant and constant terms, $a_0 = \lambda a_d$ and $a_d = \lambda a_0$. Hence, $a_0 = \lambda^2 a_0$, implying $\lambda = \pm 1$. Therefore $P^* = \pm P$. Since P is assumed nonreciprocal, $P^* \neq P$, implying $P^* = -P$. Since $P^*(1) = P(1) = -P(1)$, we would have $P(1) = 0$. Contradiction.

Rewriting trails

Now consider a reciprocal algebraic integer

$$\beta \in (1, (1+\sqrt{5})/2).$$

Two functions characterize the same “object” β :

$$P_\beta(x) \qquad \text{minimal polynomial}$$

and

$$f_\beta(z) \qquad \text{denom. of } \zeta\beta(z).$$

A priori they have nothing in common. The Parry Upper function has a lenticulus of zeroes containing $z = 1/\beta$. By a rewriting process, in base β, we show that, in a certain angular sector, approximately $(-\pi/18, +\pi/18)$, the lenticular zeroes should also be zeroes of the minimal polynomial $P_\beta(x)$.

The minorant of $\mathrm{M}(\beta)$ we are looking for arises from this subset of Galois conjugates of $1/\beta$, (therefore of β).

What is a rewriting trail ?

Let us construct the rewriting trail from "S_s" (a section of $f_\beta(z)$) to "P_β", at γ_s^{-1}.

The starting point is the identity $1 = 1$, to which we add $0 = S_{\gamma_s}(\gamma_s^{-1})$ in the (rhs) right hand side. Then we define the rewriting trail from the Rényi γ_s^{-1}-expansion of 1

$$1 = 1 + S_{\gamma_s}(\gamma_s^{-1}) = t_1\gamma_s^{-1} + t_2\gamma_s^{-2} + \ldots + t_{s-1}\gamma_s^{-(s-1)} + t_s\gamma_s^{-s} \tag{3}$$

(with $t_1 = 1, t_2 = t_3 = \ldots = t_{n-1} = 0, t_n = 1$, etc) to

$$-a_1\gamma_s^{-1} - a_2\gamma_s^{-2} + \ldots - a_{d-1}\gamma_s^{-(d-1)} - \gamma_s^{-d} = 1 - P_\beta(\gamma_s^{-1}), \tag{4}$$

by “restoring” the digits of $1-P_\beta(X)$ one after the other, from the left. We obtain a sequence $(A'_q(X))_{q\geq 1}$ of rewriting polynomials involved in this rewriting trail ; for $q\geq 1$, $A'_q\in\mathbb{Z}[X]$, $\deg(A'_q)\leq q$ and $A'_q(0)=1$. At the first step we add $0=-(-a_1-t_1)\gamma_s^{-1}S^*_{\gamma_s}(\gamma_s^{-1})$; and we obtain

$$1=-a_1\gamma_s^{-1}$$

$$+(-(-a_1-t_1)t_1+t_2)\gamma_s^{-2}+(-(-a_1-t_1)t_2+t_3)\gamma_s^{-3}+\dots$$

so that the height of the polynomial

$$(-(-a_1-t_1)t_1+t_2)X^2+(-(-a_1-t_1)t_2+t_3)X^3+\dots$$

is $\leq H+2$.

At the second step we add $0 = -(-a_2 - (-(-a_1 - t_1)t_1 + t_2))\gamma_s^{-2} S^{*}_{\gamma_s}(\gamma_s^{-1})$. Then we obtain

$$1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2}$$
$$-[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]\gamma_s^{-3} + \ldots$$

where the height of the polynomial

$$-[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]X^3 + \ldots$$

is $\leq H + (H+2) + (H+2) = 3H + 4$. Iterating this process d times we obtain

$$1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} - \ldots - a_d \gamma_s^{-d}$$
$$+ \ \textit{polynomial remainder in } \gamma_s^{-1}.$$

Denote by $V(\gamma_s^{-1})$ this polynomial remainder in γ_s^{-1}, for some $V(X) \in \mathbb{Z}[X]$, and X specializing in γ_s^{-1}. If we denote the upper bound of the height of the polynomial remainder $V(X)$, at step q, by $\lambda_q H + v_q$, we readily deduce : $v_q = 2^q$, and $\lambda_{q+1} = 2\lambda_q + 1$, $q \geq 1$, with $\lambda_1 = 1$; then $\lambda_q = 2^q - 1$.

To summarize, the first rewriting polynomials of the sequence $(A'_q(X))_{q\geq 1}$ involved in this rewriting trail are

$$A'_1(X) = -1 - (-a_1 - t_1)X,$$

$$A'_2(X) = -1 - (-a_1 - t_1)X - (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2, \quad \text{etc.}$$

For $q \geq \deg(P_\beta)$, all the coefficients of P_β are “restored” ; denote by $(h_{q,j})_{j=0,1,\ldots,s-1}$ the s-tuple of integers produced by this rewriting trail, at step q. It is such that

$$A'_q(\gamma_s^{-1})S^*_{\gamma_s}(\gamma_s^{-1}) = -P(\gamma_s^{-1}) + \gamma_s^{-q-1}\Big(\sum_{j=0}^{s-1} h_{q,j}\gamma_s^{-j}\Big). \tag{5}$$

Then take $q = d$. The (lhs) left-and side of (5) is equal to 0. Thus

$$P(\gamma_s^{-1}) = \gamma_s^{-d-1}\Big(\sum_{j=0}^{s-1} h_{d,j}\gamma_s^{-j}\Big) \qquad \Longrightarrow \qquad P(\gamma_s) = \sum_{j=0}^{s-1} h_{d,j}\gamma_s^{-j-1}.$$

The height of the polynomial

$$W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1} \qquad \text{is} \qquad \leq (2^d - 1)H + 2^d, \tag{6}$$

and is independent of $s \geq W_v$.

For any $s \geq W_v$, let us observe that $-P_\beta(\gamma_s^{-1})$ is > 0, and that the sequence $(\gamma_s^{-1})_s$ is decreasing. By an easy Lemma, the polynomial function $x \to P_\beta(x)$ is positive on $(0, \beta^{-1})$, vanishes at β^{-1}, and changes its sign for $x > \beta^{-1}$, so that $P_\beta(\gamma_s^{-1}) < 0$. We have : $\lim_{s\to\infty} P_\beta(\gamma_s^{-1}) = P_\beta(\beta^{-1}) = 0$.

to allow Galois conjugation of $1/\beta$ we need to control the remaining sums after the rewriting trails.

This is made possible using Kala-Vara's Theorem, and the fact that the irreducible factors $C(x)$, in the factorization of any $P \in \mathscr{C}$, never vanish on the unit circle.

Let us recall the definitions. The $(\delta, \mathscr{A})$-representations for a given $\delta \in \mathbb{C}$, $|\delta| > 1$ and a given alphabet $\mathscr{A} \subset \mathbb{C}$ finite, are expressions of the form $\sum_{k \geq -L} a_k \delta^{-k}$, $a_k \in \mathscr{A}$, for some integer L. We denote

$$\mathrm{Per}_{\mathscr{A}}(\delta) := \{x \in \mathbb{C} : x \text{ has an eventually periodic } (\delta, \mathscr{A})\text{−representation}\}.$$

Theorem (Kala - Vavra)

Let $\delta \in \mathbb{C}$ be an algebraic number of degree d, $|\delta| > 1$, and $a_d x^d - a_{d-1} x^{d-1} - \ldots - a_1 x - a_0 \in \mathbb{Z}[x]$, $a_0 a_d \neq 0$, be its minimal polynomial. Suppose that $|\delta'| \neq 1$ for any conjugate δ' of δ, Then there exists a finite alphabet $\mathscr{A} \subset \mathbb{Z}$ such that

$$\mathbb{Q}(\delta) = \mathrm{Per}_{\mathscr{A}}(\delta).$$

Dobrowolski-type minoration

Denote by $a_{\max} = 5.87433\ldots$ the abscissa of the maximum of the function $a \to \kappa(1,a) := \frac{1-\exp(\frac{-\pi}{a})}{2\exp(\frac{\pi}{a})-1}$ on $(0,\infty)$. Let $\kappa := \kappa(1,a_{\max}) = 0.171573\ldots$ be the value of the maximum. Let $S := 2\arcsin(\kappa/2) = 0.171784\ldots$. Denote

$$\Lambda_r\mu_r := \exp\Big(\frac{-1}{\pi}\int_0^S \mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2})-\sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{4}\Big]dx\Big)$$

$$= 1.15411\ldots, \qquad \text{a value slightly below Lehmer's number } 1.17628\ldots \tag{7}$$

Theorem (Dobrowolski type minoration)

Let α be a nonzero reciprocal algebraic integer which is not a root of unity such that $\mathrm{dyg}(\alpha) \geq 260$, with $\mathrm{M}(\alpha) < 1.176280\ldots$. Then

$$\mathrm{M}(\alpha) \geq \Lambda_r\mu_r - \Lambda_r\mu_r\,\frac{S}{2\pi}\Big(\frac{1}{\mathrm{Log}\,(\mathrm{dyg}(\alpha))}\Big) \tag{8}$$

$$\mathrm{M}(\beta) \geq 1.15411\ldots - \Lambda_r \mu_r \frac{S}{2\pi}\left(\frac{1}{\mathrm{Log}\,(n)}\right) \qquad (9)$$

n very large, i.e. β very close to 1 : lower bound $1.15411 - \varepsilon$

$\beta \geq \theta_{259}^{-1} \Longrightarrow \mathrm{M}(\beta) \geq \theta_{259}^{-1}$

$\rightarrow$ Lehmer's Cj true.

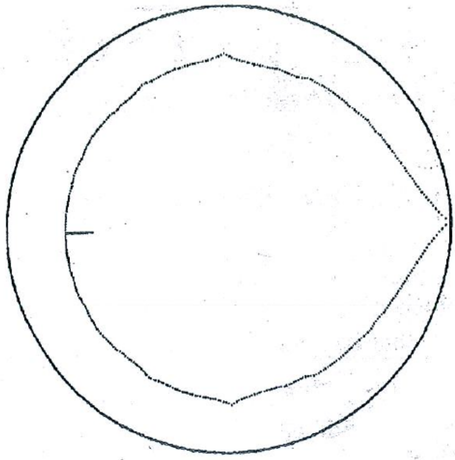

FIGURE: Solomyak's fractal.

Collab. : D. Dutykh

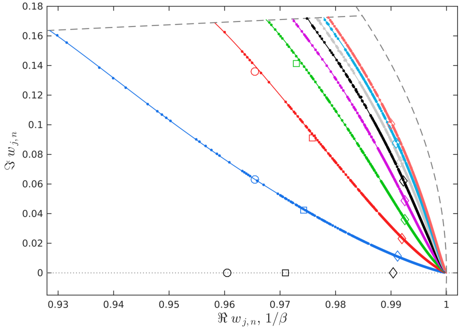

FIGURE: Curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class $\mathscr{C}$. These (universal) curves are continuous. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies inside the boundary of Solomyak's fractal.The dashed lines represent the unit circle and the top boundary of the angular sector $|\arg z| < \pi/18$. The complete set of curves, i.e. the locus of lenticuli, is obtained by symmetrization with respect to the real axis.

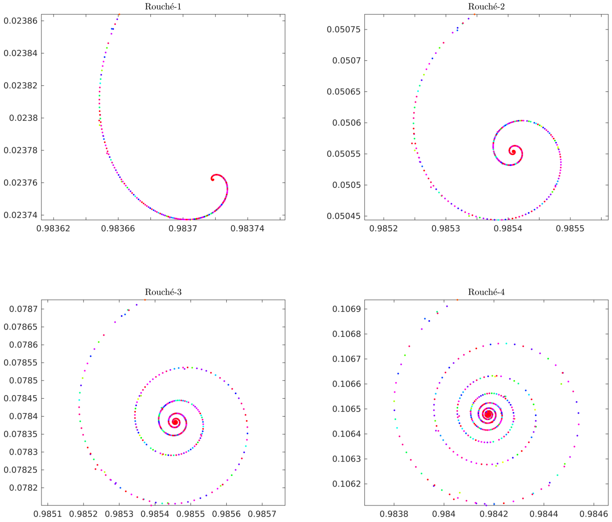

FIGURE: Curves of the lenticular jth-zeroes, containing the lenticular Galois conjugates $j = 1, 2, 3, 4$, $n = 220$)

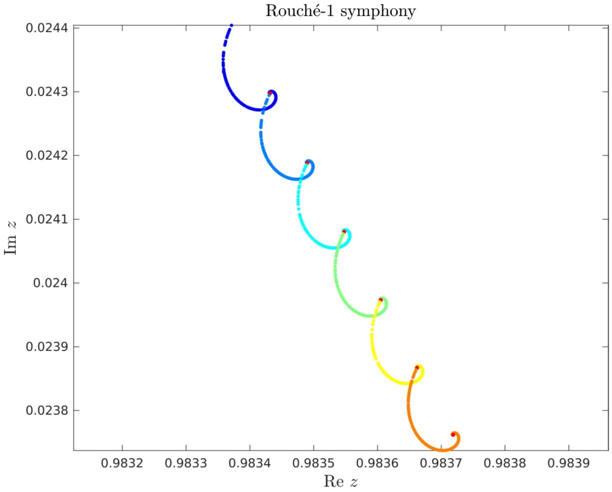

FIGURE: Curve of the lenticular first-zeroes ($j = 1$, $n = 215$ to 220)

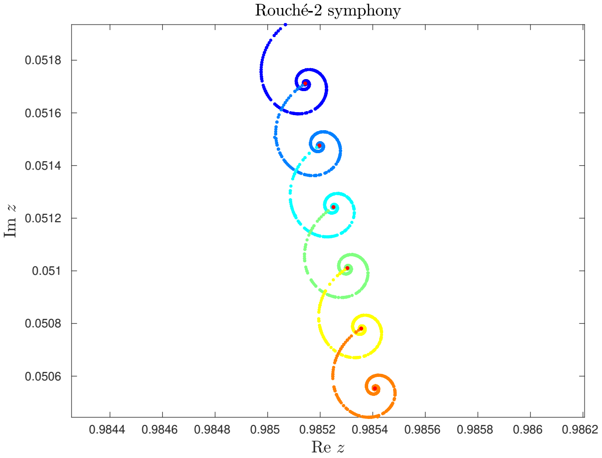

FIGURE: Curve of the lenticular 2nd-zeroes ($j = 2$, $n = 215$ to 220)