Values of certain binary partition functions represented by sum of three squares

Maciej Ulas (joint work with Bartosz Sobolewski)

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In 1798 Legendre proved that if $N$ is a positive integer and 

$$N = x^2 + y^2 + z^2$$

for some $x, y, z \in \mathbb{Z}$, then $N$ is not of the form $4^k(8s + 7)$ for $k, s \in \mathbb{N}$. In particular, the natural density of the set of integers which can not be represented by sum of three squares is equal to $1/6$. 

This raises an interesting question whether, for a given sequence of integers $(u_n)_{n \in \mathbb{N}}$, there are infinitely many solutions of the Diophantine equation 

$$(1) \quad u_n = x^2 + y^2 + z^2.$$ 

It is clear to characterize the solutions of (1) it is necessary to have a good understanding of the 2-adic behavior, or to be more precise the 2-adic valuation, of the terms of the sequence $(u_n)_{n \in \mathbb{N}}$. 

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Especially interesting is the case, when $u_n$ has a combinatorial meaning. The equation (1) with $u_n = \binom{2n}{n}$ was investigated by Granville and Zhu. They characterized those $n \in \mathbb{N}$ such that (1) has a solution in $x, y, z$. The obtained characterization is equivalent with the existence of certain patterns in (unique) binary expansion of $n$. 

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In particular, the set of integers \( n \), for which \( \binom{2n}{n} \) can be represented as a sum of three squares, has asymptotic density \( 7/8 \) in the set of all natural number. The cited authors obtained also characterization of those \( n \) such that (1) with \( u_n = n! \) has no solutions. A different approach, via automatic sequences, to this problem was presented by Deshouillers and Luca. They showed that if

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S = \{ n : \, n! \neq x^2 + y^2 + z^2 \}
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then

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S(x) = \#\{ n \leq x : \, n \in S \} = \frac{7}{8}x + O(x^{2/3}).
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This result was improved by Hajdu and Papp to

$$ S(x) = 7/8x + O(x^{1/2} \log^2 x) $$

and recently by Burns to

$$ S(x) = 7/8x + O(x^{1/2}). $$
We follow the same line of research and consider the equation (1) with $u_n = b(n)$ being binary partition function. More precisely, let $b(n)$ counts the number of partitions of $n$ with parts being powers of two. For example, $b(4) = 4$ because

$$4 = 2^2 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1$$

are all possible representations of 4 as a sum of powers of two. The sequence $(b(n))_{n \in \mathbb{N}}$ was already introduced by Euler.
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Recall that the ordinary generating function of the sequence \((b(n))_{n \in \mathbb{N}}\) has the form

\[
B(x) = \prod_{n=0}^{\infty} \frac{1}{1 - x^{2^n}} = \sum_{n=0}^{\infty} b(n)x^n.
\]

As a consequence we see that \( B(x) \) satisfies the functional equation \((1 - x)B(x) = B(x^2)\). Comparing coefficients on both sides we get that the sequence \((b(n))_{n \in \mathbb{N}}\) satisfies the recurrence: \( b(0) = b(1) = 1 \) and

\[
b(2n) = b(2n - 1) + b(n), \quad b(2n + 1) = b(2n).
\]
The corresponding series

\[ T(x) = \frac{1}{B(x)} = \prod_{n=0}^{\infty} \left(1 - x^{2^n}\right) = \sum_{n=0}^{\infty} t_n x^n \]

is the ordinary generating function for the famous Prouhet-Thue-Morse sequence \((t_n)_{n \in \mathbb{N}}\) (the PTM sequence for short). Recall that \(t_n = (-1)^{s_2(n)}\), where \(s_2(n)\) is the number of 1’s in the unique expansion of \(n\) in base 2. Equivalently, we have \(t_0 = 1\) and

\[
t_{2n} = t_n, \quad t_{2n+1} = -t_n, \quad n \geq 0.
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\[
    t_{2n} = t_n, \quad t_{2n+1} = -t_n, \quad n \geq 0.
\]

Moreover, for \(n \geq 2\), the 2-adic valuation of \(b(n)\) is equal to

\[
    \nu_2(b(n)) = \frac{1}{2} |t_n - 2t_{n-1} + t_{n-2}|.
\]

In particular, if \(n \geq 2\), then \(\nu_2(b(n)) \in \{1, 2\}\) or to be more precise,

\[
    b(n) \equiv 0 \pmod{4} \iff \nu_2(n) \equiv 0 \pmod{2} \text{ or } \nu_2(n-1) \equiv 0 \pmod{2}. \quad (2)
\]
For $m \in \mathbb{N}_+$ we define $b_m(n)$ as a convolution of $m$ copies of $b(n)$. More precisely,

$$b_m(n) = \sum_{i_1 + \ldots + i_m = n} b(i_1) \cdots b(i_m).$$

Note that $b_1(n) = b(n)$. The number $b_m(n)$ has also a combinatorial interpretation. Indeed, $b_m(n)$ is the number of binary partitions of $n$, where each part has one of $m$ possible colors.
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For \( m = 2^k - 1 \) we have that \( \nu_2(b_m(n)) \in \{1, 2\} \) for \( n \geq 2^k \).

**Theorem 1**

Let \( k \in \mathbb{N}_+ \). For \( n, i \in \mathbb{N} \) such that \( i < 2^{k+2} \) we have

\[
\nu_2(b_{2^k-1}(2^{k+2}n + i)) = \begin{cases} 
\nu_2(b(8n)) & \text{if } 0 \leq i < 2^k, \\
1 & \text{if } 2^k \leq i < 2^{k+1}, \\
2 & \text{if } 2^{k+1} \leq i < 3 \cdot 2^k, \\
1 & \text{if } 3 \cdot 2^{k+1} \leq i < 2^{k+2}.
\end{cases}
\]

In particular, \( \nu_2(b_{2^k-1}(n)) \in \{0, 1, 2\} \) and \( \nu_2(b_{2^k-1}(n)) = 0 \) if and only if \( n < 2^k \).
The case $m = 1$

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From Gauss-Legendre’s theorem and 2-adic properties of \( b(n) \) we need to understand the behaviour of the sequence \( b(n) \) (mod 32). From the equality \( b(2n + 1) = b(2n) \) it is enough to consider \( b(2n) \) (mod 32). We thus put \( u(n) := b(2n) \) and observe that

\[ u(2n) = u(2n - 1) + u(n), \quad u(2n + 1) = u(2n - 1) + 2u(n). \quad (3) \]
Proposition 2

For all \( n > 0 \) we have

\[
\nu_2(u(n)) = \begin{cases} 
1 & \text{if } \nu_2(n) \equiv 0 \pmod{2}, \\
2 & \text{if } \nu_2(n) \equiv 1 \pmod{2}.
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Lemma 3

For each \( k, n \in \mathbb{N} \) we have

\[
u_2(u(2^{2k+1}(2n+1))) \equiv u(2(2n+1)) \pmod{32}.
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Proof: The Gupta-Rödseth theorem says that

\[
b(2^{s+2} n) \equiv b(2^s n) \pmod{2^{\mu(s)}},
\]

where \( \mu(s) = \lfloor \frac{3s+4}{2} \rfloor \). Replacing \( s \) by \( 2k \) and \( b(2^{s+2} n) \) by \( u(2^{s+1} n) \), and noting that \( \mu(2k) \geq 5 \) for \( k \in \mathbb{N}_+ \) we get the statement of our lemma.
Theorem 4

Let

\[ j(n) = \frac{u(4n + 2)}{4} \mod 8, \]
\[ k(n) = \frac{u(2n + 1)}{2} \mod 8. \]

Then the sequences \((j(n))_{n \in \mathbb{N}}\) and \((k(n))_{n \in \mathbb{N}}\) are 2-automatic. More precisely, for all \(n \in \mathbb{N}\) we have

\[ j(2n) = 4 - 3t_n, \quad (4) \]
\[ j(2n + 1) = 4 + t_n, \quad (5) \]

and

\[ k(2n) = 4 - 3t_n, \quad k(2n + 1) = 4 - t_n, \]

where \(t_n\) is the \(n\) term in the PTM sequence.
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where \(t_n\) is the \(n\) term in the PTM sequence.

Proof: Examine the behaviour of \(u(n) \mod 32\).
Theorem 5

For each $a \in \{1, 3, 5, 7\}$ let $c_a = (c_a(m))_{m \in \mathbb{N}}$ be the increasing sequence such that

$$\{n \in \mathbb{N} : j(n) = a\} = \{c_a(m) : m \in \mathbb{N}\}.$$

Then the sequence $c_a$ is 2-regular. More precisely, we have

- $c_1(m) = 4m - t_m + 1$,
- $c_3(m) = 4m + t_m + 2$,
- $c_5(m) = 4m - t_m + 2$,
- $c_7(m) = 4m + t_m + 1$. 

Corollary 6

The number $b(2^n)$ is not a sum of three squares if and only if

$$n = 2^{2k-1} (8s + 2t + 3)$$

for some $k, s \in \mathbb{N}$. 

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The case $m = 3$

To get required characterization of $S_3$, we need to understand of the behaviour of $b_3(16n + i) \mod 32$ for $i = 0, 1, 2, 3, 8, 9, 10, 11$. 

Lemma 7

The following congruences holds:

\[ b_3(8n + i + 4) \equiv 2(2i + 1 + 4(-1)^n) \mod 32, \]
\[ b_3(32n + i) \equiv b_3(8n + i) \mod 64, \]
\[ b_3(8(2n + 1) + i) \equiv 4(3 + 3i - i^2 - 2(-1)^n + i) \mod 32 \]
\[ \equiv \begin{cases} 
4(3 - 2(-1)^n) \mod 32 & \text{if } i = 0, \\
4(5 + 2(-1)^n) \mod 32 & \text{if } i = 1, \\
4(5 - 2(-1)^n) \mod 32 & \text{if } i = 2, \\
4(3 + 2(-1)^n) \mod 32 & \text{if } i = 3. 
\end{cases} \]

In particular, for each $k \in \mathbb{N}^+$ and $i \in \{0, 1, 2, 3\}$, we have

\[ b_3(2^{2k}(2n + 1) + i) \equiv 2 \mod 4, \]
\[ b_3(2^{2k+1}(2n + 1) + i) \equiv b_3(8(2n + 1) + i) \mod 32. \]
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\begin{align*}
b_3(8n + i + 4) & \equiv 2(2i + 1 + 4(-1)^n)t_n \pmod{32}, \\
b_3(32n + i) & \equiv b_3(8n + i) \pmod{64}, \quad i = 0, 1, 2, 3, 4 \\
b_3(8(2n + 1) + i) & \equiv 4(3 + 3i - i^2 - 2(-1)^{n+i})t_n \pmod{32} \\
\end{align*}
\]

\[
\begin{aligned}
&\equiv \begin{cases} 
4(3 - 2(-1)^n)t_n \pmod{32} & \text{if } i = 0, \\
4(5 + 2(-1)^n)t_n \pmod{32} & \text{if } i = 1, \\
4(5 - 2(-1)^n)t_n \pmod{32} & \text{if } i = 2, \\
4(3 + 2(-1)^n)t_n \pmod{32} & \text{if } i = 3.
\end{cases}
\end{aligned}
\]

In particular, for each $k \in \mathbb{N}_+$ and $i \in \{0, 1, 2, 3\}$, we have

\[
\begin{align*}
b_3(2^{2k}(2n + 1) + i) & \equiv 2 \pmod{4}, \\
b_3(2^{2k+1}(2n + 1) + i) & \equiv b_3(8(2n + 1) + i) \pmod{32},
\end{align*}
\]
Theorem 8

We have that $n \in S_3$ if and only if

$$n = 2^{2k+1} \left( 8p + 2 \left\lfloor \frac{i}{2} \right\rfloor + 3 + 2(-1)^i t_p \right) + i$$

for some $i \in \{0, 1, 2, 3\}$ and $k \in \mathbb{N}_+$, $p \in \mathbb{N}$. 

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Values of certain binary partition functions
Theorem 8

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for some $i \in \{0, 1, 2, 3\}$ and $k \in \mathbb{N}_+, \, p \in \mathbb{N}$.

Proof: From the characterization of the 2-adic valuation of $b_3(n)$ and Lemma 7 we know that if $n \in S_3$, then necessary we have $n \pmod{16} \in \{0, 1, 2, 3, 8, 9, 10, 11\}$. Then we use case by case analysis and get the result.
The case $m = 2^k - 1, k \geq 3$

To analyze the general case we express $(b_{2k-1}(n))_{n \in \mathbb{N}}$ as the convolution of $(b_{2k}(n))_{n \in \mathbb{N}}$ and the PTM sequence, and use the following lemma.

**Lemma 9**

*For all $k, n \in \mathbb{N}$ we have*

$$b_{2k}(n) \equiv \binom{2^k}{n} + 2^{k+1} \binom{2^k - 2}{n - 2} \pmod{2^{k+2}}.$$
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\]

We consider two cases. If \( n < 2^k \), we have \( \nu_2(b_{2k-1}(n)) = 0 \). It is thus sufficient for our purposes to describe \( b_{2k-1}(n) \) modulo 8.

**Proposition 10**

*Let \( k \geq 3 \) and \( n < 2^k \). Then*

\[
b_{2k-1}(n) \equiv t_n \cdot \begin{cases} 
1 \pmod{8} & \text{if } 0 \leq n < 2^{k-2}, \\
5 \pmod{8} & \text{if } 2^{k-2} \leq n < 2^{k-1}, \\
7 \pmod{8} & \text{if } 2^{k-1} \leq n < 3 \cdot 2^{k-2}, \\
3 \pmod{8} & \text{if } 3 \cdot 2^{k-2} \leq n < 2^k.
\end{cases}
\]
As an immediate corollary, we can describe \( n < 2^k \) such that \( b_{2^k-1}(n) \) is (not) a sum of three squares.

**Corollary 11**

Let \( k \geq 3 \) and \( n < 2^k \). Then \( b_{2^k-1}(n) \) is not a sum of three squares of integers if and only if one of the following cases holds:

- \( 0 \leq n < 2^{k-2} \) and \( t_n = -1 \);
- \( 2^{k-1} \leq n < 3 \cdot 2^{k-2} \) and \( t_n = 1 \).
As an immediate corollary, we can describe $n < 2^k$ such that $b_{2^{k-1}}(n)$ is (not) a sum of three squares.

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Let $k \geq 3$ and $n < 2^k$. Then $b_{2^{k-1}}(n)$ is not a sum of three squares of integers if and only if one of the following cases holds:

- $0 \leq n < 2^{k-2}$ and $t_n = -1$;
- $2^{k-1} \leq n < 3 \cdot 2^{k-2}$ and $t_n = 1$.

We move on to the case $n \geq 2^k$. This time we have $\nu_2(b_{2^{k-1}}(n)) \in \{1, 2\}$, and by Theorem 1 we need to consider $b_{2^{k-1}}(n)$ modulo 32.

**Lemma 12**

1. For all $k, n \in \mathbb{N}$ such that $n \leq 2^k$ we have

$$\nu_2 \left( \binom{2^k}{n} \right) = k - \nu_2(n). \quad (6)$$

2. For all $m, n \in \mathbb{N}$ we have

$$\binom{2m}{2n} \equiv \binom{m}{n} \pmod{2^{\nu_2(m)+1}}. \quad (7)$$
We are now ready to describe $b_{2k-1}(n)$ modulo 32 for $n \geq 2^k$. This time, the characterization involves two terms of the PTM sequence.

**Theorem 13**

Fix $k, i, j \in \mathbb{N}$ such that $k \geq 3$, $i < 8$, and $j < 2^{k-3}$. Then for all $m \geq 1$ we have

$$b_{2k-1}(2^k m + 2^{k-3} i + j) \equiv t_j(c_i t_m + d_i t_{m-1}) \pmod{32},$$

where the coefficients $c_i, d_i$ do not depend on $k$ and are given in Table 1.

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_i$</td>
<td>1</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>−1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$d_i$</td>
<td>−5</td>
<td>−3</td>
<td>1</td>
<td>−9</td>
<td>−5</td>
<td>−3</td>
<td>−7</td>
<td>−1</td>
</tr>
</tbody>
</table>

Table: The coefficients $c_i$, $d_i$. 

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**Proof:** Consider first the case $k \geq 4$. By Lemma 9 we have

$$b_{2k-1}(n) = \sum_{l=0}^{n} b_{2k}(l) t_{n-l} \equiv \sum_{l=0}^{n} \left(\begin{array}{c} 2^k \\ l \end{array}\right) t_{n-l} \pmod{32}.$$
**Proof:** Consider first the case $k \geq 4$. By Lemma 9 we have

\[ b_{2k-1}(n) = \sum_{l=0}^{n} b_{2k}(l) t_{n-l} \equiv \sum_{l=0}^{n} \binom{2^k}{l} t_{n-l} \pmod{32}. \]

Now, by (6), the binomial coefficients with $v_2(l) < k - 4$ vanish modulo 32. Hence, assuming that $n \geq 2^k$, the above sum simplifies to

\[ b_{2k-1}(n) \equiv \sum_{l=0}^{16} \binom{2^k}{2^{k-4}l} t_{n-2^k-4l} \equiv \sum_{l=0}^{16} \binom{16}{l} t_{n-2^k-4l} \pmod{32}, \]

where the second congruence follows from (7).
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$$b_{2k-1}(n) = \sum_{l=0}^{n} b_{2k}(l) t_{n-l} \equiv \sum_{l=0}^{n} \binom{2^k}{l} t_{n-l} \pmod{32}.$$ 

Now, by (6), the binomial coefficients with $v_2(l) < k - 4$ vanish modulo 32. Hence, assuming that $n \geq 2^k$, the above sum simplifies to

$$b_{2k-1}(n) \equiv \sum_{l=0}^{16} \binom{2^k}{2^{k-4}l} t_{n-2^k-4l} \equiv \sum_{l=0}^{16} \binom{16}{l} t_{n-2^k-4l} \pmod{32},$$

where the second congruence follows from (7).

Furthermore, we can get rid of the terms with $l$ odd, since there is an even number of them and they are all congruent to 16 modulo 32. Therefore, we get the congruence

$$b_{2k-1}(n) \equiv \sum_{l=0}^{8} \binom{16}{2l} t_{n-2^k-3l} \pmod{32}.$$ 

In order to simplify the right-hand side, consider $b_{2k-1}$ at $n = 2^k m + 2^{k-3} i + j$, where $m \geq 1$, $0 \leq i < 8$, and $0 \leq j < 2^{k-3}$. 

Maciej Ulas (joint work with Bartosz Sobolewski)  
Values of certain binary partition functions
By the recurrences defining the PTM sequence, we get

\[ t_{2^k m + 2^{k-3} i + j - 2^{k-3} l} = t_j t_{8^m + i - l} = t_j \cdot \begin{cases} t_n t_{i - l} & \text{if } l \leq i, \\ -t_{n-1} t_{l - i} & \text{if } l > i. \end{cases} \]
By the recurrences defining the PTM sequence, we get

\[ t_{2^k m + 2^k - 3 i + j - 2^k - 3 l} = t_j t_{8m + i - l} = t_j \cdot \begin{cases} 
  t_n t_{i-l} & \text{if } l \leq i, \\
  -t_{n-1} t_{l-i} & \text{if } l > i.
\end{cases} \]

Hence, the claimed formula holds with the coefficients

\[ c_i = \sum_{l=0}^{i} \binom{16}{2l} t_{i-l}, \quad d_i = -\sum_{l=i+1}^{8} \binom{16}{2l} t_{l-i}, \]

and a direct computation (modulo 32) gives their values as in Table 1.
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and a direct computation (modulo 32) gives their values as in Table 1.

**Corollary 14**

For each \( k \geq 3 \) and \( n \geq 2^k \) the term \( b_{2^k-1}(n) \) is not a sum of three squares of integers if and only if \( t_n = t_{n-2^k} = 1 \). Equivalently, \( n \) is of the form

\[ n = 2^k m + l, \]

where \( l, j \in \mathbb{N} \) are such that \( t_m = t_l, \nu_2(m) \equiv 1 \pmod{2} \) and \( 0 \leq l < 2^k \).
For real $x \geq 0$ and $m \in \mathbb{N}_+$ let

$$S_m(x) = S_m \cap [0, x] = \#\{n \leq x : b_m(n) \text{ is not a sum of three squares}\}.$$
For real $x \geq 0$ and $m \in \mathbb{N}_+$ let

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Using the descriptions of the sets $S_{2k-1}$ obtained in the previous sections for various $k$ it is easy to check that

$$S_{2k-1}(x) = d_k x + O(\log x),$$

where $d_1 = d_2 = 1/12$ and $d_k = 1/6$ for $k \geq 3$. 
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In the following three results we provide more precise bounds for $S_{2k-1}(x) - d_k x$ in the case $k = 1$, $k = 2$ and $k \geq 3$, respectively. In particular, each lower and upper bound is of the form $C_1 \log_2 x + C_2$, where the constant $C_1$ is optimal.
Theorem 15

For every $x \geq 1$ we have

$$-2 < S_1(x) - \frac{x}{12} < \frac{1}{2} \log_2 x.$$

In particular, the natural density of the set $S_1$ in $\mathbb{N}$ exists and is equal to

$$\lim_{x \to +\infty} \frac{S_1(x)}{x} = \frac{1}{12}.$$

Moreover, there exists an increasing sequence $(m_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that

$$S_1(m_l) - \frac{m_l}{12} \sim \frac{1}{2} \log_2 m_l.$$
Proof: For $x \in \mathbb{R}$ define

$$P(x) = \# \{ s \in \mathbb{N} : 8s + 2t_s + 3 \leq x \}, \quad Q(x) = \sum_{k=0}^{\infty} P \left( \frac{x}{4^k} \right).$$
Proof: For $x \in \mathbb{R}$ define
\[ P(x) = \# \{ s \in \mathbb{N} : 8s + 2ts + 3 \leq x \}, \quad Q(x) = \sum_{k=0}^{x} P \left( \frac{x}{4^k} \right). \]

We have that that $Q \left( \frac{x}{2} \right) = \# \{ n \leq x : b(2n) \in S \}$, hence by the relation $b(2n + 1) = b(2n)$, we get
\[ S(x) = Q \left( \frac{x}{4} \right) + Q \left( \frac{x - 1}{4} \right). \]

For $m \in \mathbb{N}$ and $i = 0, 1, 2, 3$ we have the recurrence relations
\[ Q(4m + i) = Q(m) + P(4m + i). \]
Proof: For $x \in \mathbb{R}$ define

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For $m \in \mathbb{N}$ and $i = 0, 1, 2, 3$ we have the recurrence relations

$$Q(4m + i) = Q(m) + P(4m + i).$$

Also, for $i < 8$ we have

$$P(8m + i) = m + \begin{cases} 0 \quad \text{if } i = 0, \\ T_m \quad \text{if } i = 1, 2, 3, 4, \\ 1 \quad \text{if } i = 5, 6, 7. \end{cases}$$

Put

$$R(x) = Q(x) - \frac{x}{6}.$$ 

By induction on length $L(m)$ of binary expansion of $m \in \mathbb{N}_+$ we get

$$-\frac{2}{3} \leq R(m) \leq \frac{1}{4} \lfloor \log_2 m \rfloor - \frac{1}{6}. \quad (8)$$
Now, define $m_0 = 0$ and $m_{l+1} = 16m_l + 36$ for $l \in \mathbb{N}$. Using the recurrence relations above and the fact that $4 \mid m_l$, we get

$$R(m_{l+1}) = R(16m_l + 36) = R(4m_l) + 1 - T_{m_l} = R(m_l) + 1 - T_{m_l}.$$ 

By induction one can quickly prove that $T_{m_l} = 0$ for all $l \in \mathbb{N}$, and thus we get $R(m_l) = l$ and consequently $S_1(m_l) - m_l/12 = 2(l - 1)$. 
Now, define \( m_0 = 0 \) and \( m_{l+1} = 16m_l + 36 \) for \( l \in \mathbb{N} \). Using the recurrence relations above and the fact that \( 4 \mid m_l \), we get

\[
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\]

By induction one can quickly prove that \( T_{m_l} = 0 \) for all \( l \in \mathbb{N} \), and thus we get \( R(m_l) = l \) and consequently \( S_1(m_l) - m_l/12 = 2(l - 1) \).

**Theorem 16**

*For all* \( x \geq 1 \) *we have*

\[
\left| S_3(x) - \frac{x}{12} \right| \leq \frac{1}{6} \log_2 x + \frac{3}{2}.
\]

*In particular, the natural density of the set* \( S_3 \) *in* \( \mathbb{N} \) *exists and is equal to*

\[
\lim_{x \to +\infty} \frac{S_3(x)}{x} = \frac{1}{12}.
\]

*Moreover, there exist increasing sequences* \( (m_l)_{l \in \mathbb{N}}, (n_l)_{l \in \mathbb{N}} \subset \mathbb{N} \) *such that*

\[
S_3(m_l) - \frac{m_l}{12} \sim \frac{1}{6} \log_2 m_l,
\]

\[
S_3(n_l) - \frac{n_l}{12} \sim -\frac{1}{6} \log_2 n_l.
\]
Theorem 17

If \( k \geq 3 \), then for all \( x \geq 2^k \) we have

\[
\left| S_{2^{k-1}}(x) - \frac{x}{6} + 2^{k-2} \right| \leq \frac{2^{k-2}}{3} (\log_2 x - k + 17).
\]

In particular, the natural density of the set \( S_{2^{k-1}} \) in \( \mathbb{N} \) exists and is equal to

\[
\lim_{x \to +\infty} \frac{S_{2^{k-1}}(x)}{x} = \frac{1}{6}.
\]

Moreover, there exist increasing sequences \( (m_l)_{l \in \mathbb{N}}, (n_l)_{l \in \mathbb{N}} \subset \mathbb{N} \) such that

\[
S_{2^{k-1}}(m_l) - \frac{m_l}{6} \sim \frac{2^{k-2}}{3} \log_2 m_l,
\]
\[
S_{2^{k-1}}(n_l) - \frac{n_l}{6} \sim -\frac{2^{k-2}}{3} \log_2 n_l.
\]
It is natural to ask whether it is possible to obtain results concerning the representation of $b_m(n)$ as a sum of three squares for any $m \in \mathbb{N}_+$. 

**Problem 1**

*Describe the set $S_m$ for $m \in \mathbb{N}_+$.***
It is natural to ask whether it is possible to obtain results concerning the representation of $b_m(n)$ as a sum of three squares for any $m \in \mathbb{N}_+$. 

**Problem 1**

*Describe the set $S_m$ for $m \in \mathbb{N}_+$.*

The direct approach we, namely reduction modulo a power of 2, is most likely not applicable in the general case, as it seems that for all $m \neq 2^k - 1$ the valuations $\nu_2(b_m(n))$ are unbounded. In such a case one would need to compute $b_m(n) \mod 2^{\nu_2(b_m(n)) + 3}$ and we do not see how this can be done without prior knowledge of $\nu_2(b_m(n))$. Therefore, we expect that obtaining an exact description of $S_m$ for even a single value $m \neq 2^k - 1$ is hard.
We obtained precise characterization of those $n \in \mathbb{N}$ such that $b(n)$ is a sum of three squares. In particular the set of such numbers has asymptotic density equal to $11/12$. A more difficult question is whether the set

$$T_1 = \{ n \in \mathbb{N} : b(2n) = \Box + \Box \}$$

is infinite or not.
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\[
\mathcal{T}_1 = \{ n \in \mathbb{N} : b(2n) = \Box + \Box \}
\]

is infinite or not.

To get a clue what can be expected, we computed the values of \( b(2n) \) for \( n \leq 2^{20} \) and check whether \( b(2n) \) is a sum of two squares. We put

\[
\mathcal{T}_1(x) = \#(\mathcal{T}_1 \cap [0, x]).
\]
In the table below we present the values of $T(2^n)$ for $n \leq 20$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
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Our numerical computations suggest the following

**Conjecture 1**
The set $T$ is infinite.

The following heuristic reasoning provides further evidence towards our conjecture. More precisely, recall that the counting function of the sums of two squares up to $x$ is $O(x/\sqrt{\log x})$. Thus, one can say that the probability that a random positive integer $n$ can be written as a sum of two squares of integers is $c/\sqrt{\log n}$. 

Maciej Ulas (joint work with Bartosz Sobolewski)

Values of certain binary partition functions
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Since, $\log_2 b(n) \approx \frac{1}{2} (\log_2 n)^2$ one could conjecture that the expectation that $b(n)$ is a sum of two squares is $c'/\log n$ for some $c' > 0$, provided that $b(n)$ behaves like a random integer of its size. Thus, up to $x$, we would have at least

$$\sum_{n \leq x} \frac{1}{\log n} = \frac{x}{\log x} + O(x/\log^2 x)$$

values of $n$ such that $b(n)$ is a sum of two squares.
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\[
\sum_{n \leq x} \frac{1}{\log n} = \frac{x}{\log x} + O(x/\log^2 x)
\]

values of \( n \) such that \( b(n) \) is a sum of two squares.

Conjecture 2

There exists a positive real number \( c \) such that

\[
\mathcal{T}(x) = c \frac{x}{\log x} + O(x/\log^2 x)
\]

as \( x \to +\infty \).

Our computations seem to confirm such an expectation. Here are the values \( \mathcal{T}(2^m) \frac{m}{2^m} \) for \( m = 10, \ldots, 20 \).

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<th>( m )</th>
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<th>12</th>
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<td>( \mathcal{T}(2^m) \frac{m}{2^m} )</td>
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Thank you for your attention;-)