Fractal tiles induced by tent maps

Paul Surer

University of Natural Resources and Life Sciences Vienna (BOKU)
Department of Integrative Biology and Biodiversity Research
Institute of Mathematics

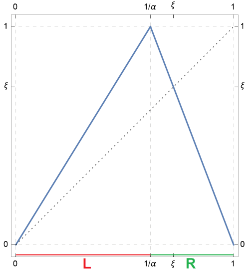
Liège, May 2023

Tent map

Definition (Tent-map)

Let $\alpha > 1$ and $\beta = \beta(\alpha) = \frac{\alpha}{\alpha - 1}$ (hence, $\alpha^{-1} + \beta^{-1} = 1$). The tent-map T_α is defined by

$$T_\beta : [0,1] \to [0,1], x \longmapsto \begin{cases} \alpha x & \text{if } x \in L := [0, \alpha^{-1}], \\ \beta(1-x) & \text{if } x \in R := (\alpha^{-1}, 1]. \end{cases}$$



Dynamical properies have been studied intensively, e.g. Lagarias et.at. (1993, 1994), Scheicher, Sirvent, S. (2016).

Program for the next 20 minutes

We are interested in geometric realisations (tiles) of tent maps for appropriate choices of α.

- Introduction: Definition of tiles
- Results: Tiling properties, Hausdorff dimension of the boundary
- Background: Relations with Rauzy fractals

Notations

Definition (Pisot number)

A real algebraic integer α is a **Pisot number** if $\alpha > 1$ and the Galois conjugates different from α are contained in the unit circle.
We call α a **special Pisot number** if both α and $\beta(\alpha)$ are Pisot numbers.

We suppose that α is a special Pisot unit (i.e., a special Pisot number that is an algebraic unit). Notations:

$d+1$	algebraic degree of α;
$\lambda_1, \ldots, \lambda_d$	Galois conjugates different from α;
$A_\alpha \in \mathbb{R}^{d \times d}$	matrix similar to $\mathtt{diag}(\lambda_1, \ldots, \lambda_d)$;
$\Psi_\alpha : \mathbb{Q}(\alpha) \longrightarrow \mathbb{R}^d$	embedding that satisfies $\Psi_\alpha(\alpha x) = A_\alpha \Psi_\alpha(x)$ for all $x \in \mathbb{Q}(\alpha)$;
$B_\alpha := A_\alpha (A_\alpha - I_d)^{-1}$	where I_d denotes the $d \times d$ identity matrix.

Tent-tiles

Recall that

$$T_\beta : [0,1] \to [0,1], x \longmapsto \begin{cases} \alpha x & \text{if } x \in L := [0, \alpha^{-1}], \\ \beta(1-x) & \text{if } x \in R := (\alpha^{-1}, 1]. \end{cases}$$

Accordingly we define

$$f_L : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \mathbf{x} \longmapsto A_\alpha \mathbf{x},$$
$$f_R : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \mathbf{x} \longmapsto B_\alpha (\Psi_\alpha(1) - \mathbf{x}).$$

Definition (Tent-tile)

The tent-tile $\mathcal{F}_\alpha$ associated with the simple Pisot unit α is the uniquely determined compact set that satisfies the set equation

$$\mathcal{F}_\alpha = f_L(\mathcal{F}_\alpha) \cup f_R(\mathcal{F}_\alpha).$$

Special Pisot numbers

Theorem (Smyth, 1999)

There exist exactly 11 *special Pisot numbers.*

Number	Approx. value	Minimal polynomial
α_{-5}	$4.0796\ldots$	x^3-5x^2+4x-1
α_{-4}	$3.62966\ldots$	$x^4-5x^3+6x^2-4x+1$
α_{-3}	$3.1479\ldots$	x^3-4x^2+3x-1
α_{-2}	$2.61803\ldots$	x^2-3x+1
α_{-1}	$2.32472\ldots$	x^3-3x^2+2x-1
α_0	2	$x-2$
α_1	$1.75488\ldots$	x^3-2x^2+x-1
α_2	$1.61803\ldots$	x^2-x-1
α_3	$1.46557\ldots$	x^3-x^2-1
α_4	$1.38028\ldots$	x^4-x^3-1
α_5	$1.32472\ldots$	x^3-x-1

They are arranged such that $\beta(\alpha_j)=\alpha_{-j}$. All but $\alpha_0=2$ are algebraic units.

Properties of tent-tiles

Theorem (Scheicher-Sirvent-S.)

All tent-tiles have a positive d-dimensional Lebesgue measure.

We further studied

- Hausdorff-dimension of the boundary
- Tiling properties: do the tent-tiles induce a lattice tiling of $\mathbb{R}^d$?

The tent-tiles show up many different behaviours and properties in view of their shape, topology, and tiling properties.

One-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

The one-dimensional tent-tiles associated with α_2 and α_{-2} are intervals.

- *Let* $\alpha = \alpha_2$ *(dominant root of* $x^2 - x - 1$*) and* $\beta = \beta(\alpha)\,(= \alpha_{-2})$*.*
  Then $\mathcal{F}_\alpha = \left[\frac{1-\sqrt{5}}{4}, \frac{1}{2}\right]$*.*

- *Let* $\alpha = \alpha_{-2}$ *(dominant root of* $x^2 - 3x + 1$*) and* $\beta = \beta(\alpha)\,(= \alpha_2)$*.*
  Then $\mathcal{F}_\alpha = \left[\frac{-1-\sqrt{5}}{2}, 0\right]$*.*

Planar tent tiles

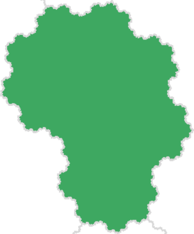
Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_1 \approx 1.75488$ *(dominant root of* $x^3 - 2x^2 + x - 1$*) and* $\beta = \beta(\alpha)$*.*

- *The collection* $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ *is a proper tiling of the Euclidean space* $\mathbb{R}^2$ *where*

$$\mathfrak{L} = \{u_1 \Psi_\alpha(\beta - \alpha) + u_2 \Psi_\alpha(\beta - \alpha^2) : u_1, u_2 \in \mathbb{Z}\}.$$

- $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.10026.$



Planar tent tiles

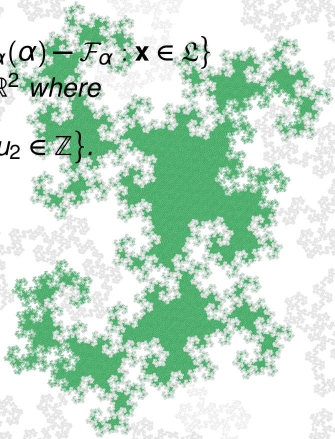
Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-1} \approx 2.32472$ (dominant root of $x^3 - 3x^2 + 2x - 1$) and $\beta = \beta(\alpha)$.

- *The collection $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\} \cup \{\mathbf{x} + \Psi_\alpha(\alpha) - \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^2$ where*

  $$\mathfrak{L} = \{u_1 \Psi_\alpha(\alpha - \beta) + u_2 \Psi_\alpha(\alpha - \beta^2) : u_1, u_2 \in \mathbb{Z}\}.$$

- $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.70018.$



Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_3 \approx 1.46557$ *(dominant root of* $x^3 - x^2 - 1$*).*

- *The collection* $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\} \cup \{\mathbf{x} + \Psi_\alpha(1) - \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ *is a proper tiling of the Euclidean space* $\mathbb{R}^2$ *where*

  $$\mathfrak{L} = \{u_1\Psi_\alpha(1-\alpha) + u_2\Psi_\alpha(1-\alpha^2) : u_1, u_2 \in \mathbb{Z}\}.$$

- $\dim_H(\partial\mathcal{F}_\alpha) \approx 1.02952.$

Planar tent tiles

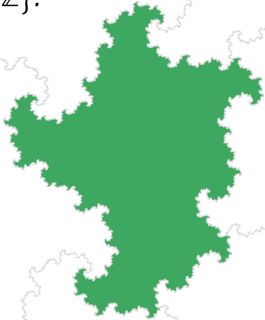
Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-3} \approx 3.1479$ (dominant root of $x^3 - 4x^2 + 3x - 1$) and $\beta = \beta(\alpha)$.

- *The collection $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\} \cup \{\mathbf{x} + \Psi_\alpha(1) - \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^2$ where*

$$\mathfrak{L} = \{u_1 \Psi_\alpha(1-\beta) + u_2 \Psi_\alpha(1-\beta^2) : u_1, u_2 \in \mathbb{Z}\}.$$

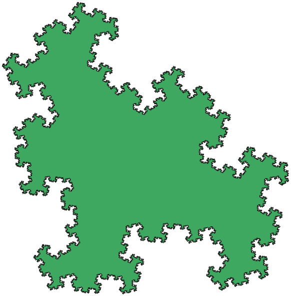
- $\dim_H(\partial\mathcal{F}_\alpha) \approx 1.25074.$



Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

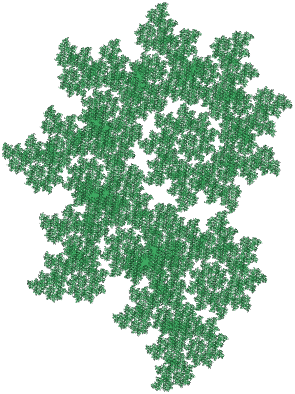
Let $\alpha = \alpha_5 \approx 1.32472$ (dominant root of $x^3 - x - 1$). Then $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.37858$.



Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-5} \approx 4.0796$ *(dominant root of* $x^3 - 5x^2 + 4x - 1$*). Then* $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.92089$*.*



Three-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_4 \approx 1.38028$ (dominant root of $x^4 - x^3 - 1$).

- *The collection $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\} \cup \{\mathbf{x} + \Psi_\alpha(1) - \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^3$ where*

$$\mathfrak{L} = \{u_1\Psi_\alpha(1-\alpha) + u_2\Psi_\alpha(1-\alpha^2) + u_3\Psi_\alpha(1-\alpha^3) : u_1, u_2, u_3 \in \mathbb{Z}\}.$$

- $\dim_H(\partial\mathcal{F}_\alpha) \le \dim_B(\partial\mathcal{F}_\alpha) \approx 2.74421.$

Three-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-4} \approx 3.62966$ (dominant root of $x^4 - 5x^3 + 6x^2 - 4x + 1$) and $\beta = \beta(\alpha)$.

- *The collection $\{\mathbf{x} + \mathcal{F}_\alpha : \mathbf{x} \in \mathfrak{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^3$ where*

$$\mathfrak{L} = \{u_1\Psi_\alpha(1-\beta) + u_2\Psi_\alpha(1-\beta^2) + u_3\Psi_\alpha(1-\beta^3) : u_1, u_2, u_3 \in \mathbb{Z}\}.$$

- $\dim_H(\partial\mathcal{F}_\alpha) \leq \dim_B(\partial\mathcal{F}_\alpha) \approx 2.815.$

Rauzy fractals

Let

- $\mathcal{A} = \{1, \dots, m\}$ be a finite set (alphabet);
- $\mathcal{A}^*$ be the set of finite words over $\mathcal{A}$;
- $\varepsilon \in \mathcal{A}$ be the empty word;
- $|W|_a$ be the number of occurrences of the $a \in \mathcal{A}$ in the word $W \in \mathcal{A}^*$;
- $\ell : \mathcal{A}^* \longrightarrow \mathbb{R}^m$ be the Ablianisation map

$$\ell(W) = (|W|_1, |W|_2, \dots, |W|_m)^T.$$

A substitution over $\mathcal{A}$ is a non-erasing morphism $\zeta : \mathcal{A}^* \longrightarrow \mathcal{A}^*$.
Define the incidence matrix
$M_\zeta := (\ell(\zeta(1)), \ell(\zeta(2)), \dots, \ell(\zeta(m))) \in \mathbb{R}^{m\times m}$ and observe that for each $W \in \mathcal{A}^*$ we have $\ell(\zeta(W)) = M_\zeta \cdot \ell(W)$.

Rauzy fractals

We require ζ to be

- primitive: there exists a positive integer n such that M_ζ^n is strictly positive;
- unimodular: $\det(M_\zeta) = \pm 1$;
- Pisot: the Perron-Frobenius eigenvalue if M_ζ is a Pisot number.

Define

- $\mathbb{K}_e$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvector of M_ζ with respect to the dominant eigenvalue;
- $\mathbb{K}_c$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvectors of M_ζ with respect to the Galois conjugates different from the dominant eigenvalue;
- $\mathbb{K}_s$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvectors of M_ζ with respect to the remaining eigenvalues (if they exist);
  Observe:
  $$\mathbb{R}^m = \mathbb{K}_e \oplus \mathbb{K}_c \oplus \mathbb{K}_s$$
  .
- $\pi : \mathbb{R}^m \longrightarrow \mathbb{K}_c$: a projection parallel to $\mathbb{K}_e$ and $\mathbb{K}_s$;
- $h : \mathbb{K}_c \longrightarrow \mathbb{K}_c$: the restriction of the action of M_ζ on $\mathbb{K}_c$, i.e. for all $\mathbf{x} \in \mathbb{R}^m$ we have
  $$\pi(M_\zeta \mathbf{x}) = h \circ \pi(\mathbf{x})$$

Rauzy fractals

There exists a uniquely determined invariant set list $(\mathcal{R}_\zeta(a) : a \in \mathcal{A}\}$ of compact sets in $\mathbb{K}_c$ that satisfies the system of set equations

$$\mathcal{R}_\zeta(a) = \bigcup_{\sigma(b)=PaS} \big(\pi \circ \ell(P) + h(\mathcal{R}_\zeta(a))\big) \text{ for } a = 1, \ldots, m.$$

The Rauzy fractal $\mathcal{R}_\zeta$ associated with ζ is the union

$$\mathcal{R}_\zeta = \bigcup_{a \in \mathcal{A}} \mathcal{R}_\zeta(a).$$

An concrete example

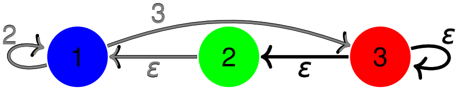
- $\mathcal{A} = \{1, 2, 3\}$;
- $\zeta : 1 \mapsto 21, 2 \mapsto 3, 3 \mapsto 31$
- $M_\zeta = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.
- $\chi_{M_\zeta}(x) = 1 - x + 2x^2 - x^3$
- 3 eigenvalues:

$$\lambda_0 \approx 1.75488 (= \alpha_1) \text{ (PF)}$$
$$\lambda \approx 0.122561 + 0.744862i$$
$$\overline{\lambda} \approx 0.122561 - 0.744862i$$

- ζ is irreducible since the algebraic degree of λ equals m;
- $\mathbb{K}_c \cong \mathbb{R}$, $\mathbb{K}_e \cong \mathbb{C}$, $\mathbb{K}_s$ is the trivial space, $h : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto \lambda \cdot z$.

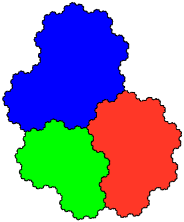
An concrete example

$\zeta : 1 \mapsto 21, 2 \mapsto 3, 3 \mapsto 31$



$$\mathcal{R}_\zeta(1) = \big(\lambda + \lambda \cdot \mathcal{R}_\zeta(1)\big) \cup \big(\lambda^2 + \lambda \cdot \mathcal{R}_\zeta(3)\big)$$
$$\mathcal{R}_\zeta(2) = \lambda \cdot \mathcal{R}_\zeta(1)$$
$$\mathcal{R}_\zeta(3) = \lambda \cdot \mathcal{R}_\zeta(2) \cup \lambda \cdot \mathcal{R}_\zeta(3)$$

$$\mathcal{R}_\zeta = \mathcal{R}_\zeta(1) \cup \mathcal{R}_\zeta(2) \cup \mathcal{R}_\zeta(3)$$

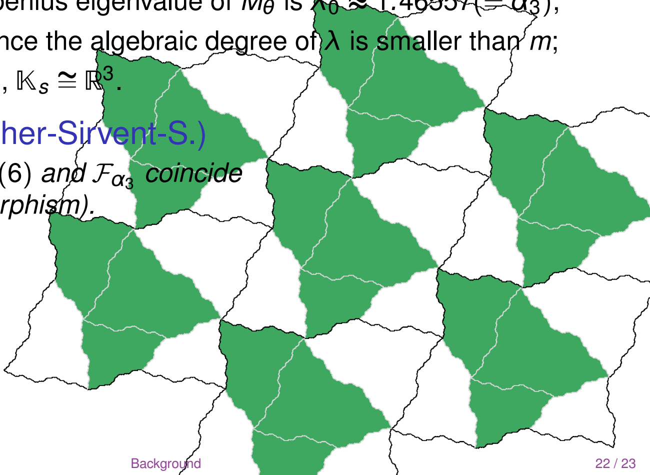


Theorem (Scheicher-Sirvent-S.)

$\mathcal{R}_\zeta$ and $\mathcal{F}_{\alpha_1}$ coincide (up to a linear isomorphism).

Another example

- $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$;
- $\theta : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 43, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 61$;
- $\chi_{M_\theta}(x) = (x^3 - x^2 - 1)(x^3 - x^2 + 1)$
- The Perron-Frobenius eigenvalue of M_θ is $\lambda_0 \approx 1.46557 (= \alpha_3)$;
- ζ is reducible since the algebraic degree of λ is smaller than m;
- $\mathbb{K}_c \cong \mathbb{R}$, $\mathbb{K}_e \cong \mathbb{C}$, $\mathbb{K}_s \cong \mathbb{R}^3$.



Theorem (Scheicher-Sirvent-S.)

$\mathcal{R}_\theta(4) \cup \mathcal{R}_\theta(5) \cup \mathcal{R}_\theta(6)$ and $\mathcal{F}_{\alpha_3}$ coincide (up to a linear isomorphism).

Thanks

Thank you for your presence!

Thank you for your attention!

MERCI!