Fractal tiles induced by tent maps

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Liège, May 2023
Tent map

Definition (Tent-map)
Let $\alpha > 1$ and $\beta = \beta(\alpha) = \frac{\alpha}{\alpha - 1}$ (hence, $\alpha^{-1} + \beta^{-1} = 1$). The tent-map $T_{\alpha}$ is defined by

$$T_{\beta} : [0, 1] \rightarrow [0, 1], x \mapsto \begin{cases} \alpha x & \text{if } x \in L := [0, \alpha^{-1}], \\ \beta(1 - x) & \text{if } x \in R := (\alpha^{-1}, 1]. \end{cases}$$

Dynamical properties have been studied intensively, e.g. Lagarias et.al. (1993, 1994), Scheicher, Sirvent, S. (2016).
We are interested in geometric realisations (tiles) of tent maps for appropriate choices of $\alpha$.

- Introduction: Definition of tiles
- Results: Tiling properties, Hausdorff dimension of the boundary
- Background: Relations with Rauzy fractals
Definition (Pisot number)

A real algebraic integer $\alpha$ is a **Pisot number** if $\alpha > 1$ and the Galois conjugates different from $\alpha$ are contained in the unit circle. We call $\alpha$ a **special Pisot number** if both $\alpha$ and $\beta(\alpha)$ are Pisot numbers.

We suppose that $\alpha$ is a special Pisot unit (i.e., a special Pisot number that is an algebraic unit). Notations:

- $d + 1$: algebraic degree of $\alpha$;
- $\lambda_1, \ldots, \lambda_d$: Galois conjugates different from $\alpha$;
- $A_\alpha \in \mathbb{R}^{d \times d}$: matrix similar to $\text{diag}(\lambda_1, \ldots, \lambda_d)$;
- $\psi_\alpha : \mathbb{Q}(\alpha) \longrightarrow \mathbb{R}^d$: embedding that satisfies $\psi_\alpha(\alpha x) = A_\alpha \psi_\alpha(x)$ for all $x \in \mathbb{Q}(\alpha)$;
- $B_\alpha := A_\alpha(A_\alpha - I_d)^{-1}$: where $I_d$ denotes the $d \times d$ identity matrix.
Recall that
\[
T_\beta : [0, 1] \rightarrow [0, 1], \ x \mapsto \begin{cases} 
\alpha x & \text{if } x \in L := [0, \alpha^{-1}], \\
\beta(1 - x) & \text{if } x \in R := (\alpha^{-1}, 1]. 
\end{cases}
\]

Accordingly we define
\[
f_L : \mathbb{R}^d \rightarrow \mathbb{R}^d, \ x \mapsto A_\alpha \ x,
\]
\[
f_R : \mathbb{R}^d \rightarrow \mathbb{R}^d, \ x \mapsto B_\alpha (\psi_\alpha(1) - x).
\]

**Definition (Tent-tile)**

The tent-tile $\mathcal{F}_\alpha$ associated with the simple Pisot unit $\alpha$ is the uniquely determined compact set that satisfies the set equation
\[
\mathcal{F}_\alpha = f_L(\mathcal{F}_\alpha) \cup f_R(\mathcal{F}_\alpha).
\]
Special Pisot numbers

Theorem (Smyth, 1999)

There exist exactly 11 special Pisot numbers.

<table>
<thead>
<tr>
<th>Number</th>
<th>Approx. value</th>
<th>Minimal polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{-5}$</td>
<td>4.0796…</td>
<td>$x^3 - 5x^2 + 4x - 1$</td>
</tr>
<tr>
<td>$\alpha_{-4}$</td>
<td>3.62966…</td>
<td>$x^4 - 5x^3 + 6x^2 - 4x + 1$</td>
</tr>
<tr>
<td>$\alpha_{-3}$</td>
<td>3.1479…</td>
<td>$x^3 - 4x^2 + 3x - 1$</td>
</tr>
<tr>
<td>$\alpha_{-2}$</td>
<td>2.61803…</td>
<td>$x^2 - 3x + 1$</td>
</tr>
<tr>
<td>$\alpha_{-1}$</td>
<td>2.32472…</td>
<td>$x^3 - 3x^2 + 2x - 1$</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>2</td>
<td>$x - 2$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>1.75488…</td>
<td>$x^3 - 2x^2 + x - 1$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1.61803…</td>
<td>$x^2 - x - 1$</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>1.46557…</td>
<td>$x^3 - x^2 - 1$</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>1.38028…</td>
<td>$x^4 - x^3 - 1$</td>
</tr>
<tr>
<td>$\alpha_5$</td>
<td>1.32472…</td>
<td>$x^3 - x - 1$</td>
</tr>
</tbody>
</table>

They are arranged such that $\beta(\alpha_j) = \alpha_{-j}$. All but $\alpha_0 = 2$ are algebraic units.
Properties of tent-tiles

Theorem (Scheicher-Sirvent-S.)

All tent-tiles have a positive $d$-dimensional Lebesgue measure.

We further studied

- Hausdorff-dimension of the boundary
- Tiling properties: do the tent-tiles induce a lattice tiling of $\mathbb{R}^d$?

The tent-tiles show up many different behaviours and properties in view of their shape, topology, and tiling properties.
One-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

The one-dimensional tent-tiles associated with $\alpha_2$ and $\alpha_{-2}$ are intervals.

Let $\alpha = \alpha_2$ (dominant root of $x^2 - x - 1$) and $\beta = \beta(\alpha) (= \alpha_{-2})$.
Then $F_{\alpha} = \left[ \frac{1 - \sqrt{5}}{4}, \frac{1}{2} \right]$.

Let $\alpha = \alpha_{-2}$ (dominant root of $x^2 - 3x + 1$) and $\beta = \beta(\alpha) (= \alpha_2)$.
Then $F_{\alpha} = \left[ \frac{-1 - \sqrt{5}}{2}, 0 \right]$. 
Planar tent tiles

**Theorem (Scheicher-Sirvent-S.)**

Let $\alpha = \alpha_1 \approx 1.75488$ (dominant root of $x^3 - 2x^2 + x - 1$) and $\beta = \beta(\alpha)$.

- The collection $\{x + \mathcal{F}_\alpha : x \in \mathcal{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^2$ where
  \[ \mathcal{L} = \{ u_1 \psi_\alpha(\beta - \alpha) + u_2 \psi_\alpha(\beta - \alpha^2) : u_1, u_2 \in \mathbb{Z} \}. \]
  
- $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.10026$. 

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Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-1} \approx 2.32472$ (dominant root of $x^3 - 3x^2 + 2x - 1$) and $\beta = \beta(\alpha)$.

- The collection $\{x + F_\alpha : x \in \mathcal{L}\} \cup \{x + \psi_\alpha(\alpha) - F_\alpha : x \in \mathcal{L}\}$
  is a proper tiling of the Euclidean space $\mathbb{R}^2$ where

  $\mathcal{L} = \{u_1 \psi_\alpha(\alpha - \beta) + u_2 \psi_\alpha(\alpha - \beta^2) : u_1, u_2 \in \mathbb{Z}\}$.

- $\dim_H(\partial F_\alpha) \approx 1.70018$. 
Theorem (Scheicher-Sirvent-S.)

Let \( \alpha = \alpha_3 \approx 1.46557 \) (dominant root of \( x^3 - x^2 - 1 \)).

- The collection \( \{ x + F_\alpha : x \in L \} \cup \{ x + \Psi_\alpha(1) - F_\alpha : x \in L \} \)
  is a proper tiling of the Euclidean space \( \mathbb{R}^2 \) where

\[
L = \{ u_1 \Psi_\alpha(1 - \alpha) + u_2 \Psi_\alpha(1 - \alpha^2) : u_1, u_2 \in \mathbb{Z} \}.
\]

- \( \dim_H(\partial F_\alpha) \approx 1.02952 \).
Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-3} \approx 3.1479$ (dominant root of $x^3 - 4x^2 + 3x - 1$) and $\beta = \beta(\alpha)$.

$\triangleright$ The collection $\{x + \mathcal{F}_\alpha : x \in \mathcal{L}\} \cup \{x + \psi_\alpha(1) - \mathcal{F}_\alpha : x \in \mathcal{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^2$ where

$\mathcal{L} = \{u_1 \psi_\alpha(1 - \beta) + u_2 \psi_\alpha(1 - \beta^2) : u_1, u_2 \in \mathbb{Z}\}$.

$\triangleright$ $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.25074$. 
Planar tent tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_5 \approx 1.32472$ (dominant root of $x^3 - x - 1$). Then $\dim_H(\partial F_\alpha) \approx 1.37858$. 
Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-5} \approx 4.0796$ (dominant root of $x^3 - 5x^2 + 4x - 1$). Then $\dim_H(\partial \mathcal{F}_\alpha) \approx 1.92089$. 

Planar tent tiles
Three-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_4 \approx 1.38028$ (dominant root of $x^4 - x^3 - 1$).

1. The collection $\{x + F_\alpha : x \in \mathcal{L}\} \cup \{x + \psi_\alpha(1) - F_\alpha : x \in \mathcal{L}\}$ is a proper tiling of the Euclidean space $\mathbb{R}^3$ where

\[ \mathcal{L} = \left\{ u_1 \psi_\alpha(1 - \alpha) + u_2 \psi_\alpha(1 - \alpha^2) + u_3 \psi_\alpha(1 - \alpha^3) : u_1, u_2, u_3 \in \mathbb{Z} \right\}. \]

2. $\dim_H(\partial F_\alpha) \leq \dim_B(\partial F_\alpha) \approx 2.74421$. 

\[ \end{document} \]
Three-dimensional tent-tiles

Theorem (Scheicher-Sirvent-S.)

Let $\alpha = \alpha_{-4} \approx 3.62966$ (dominant root of $x^4 - 5x^3 + 6x^2 - 4x + 1$) and $\beta = \beta(\alpha)$.

- **The collection** $\{x + F_\alpha : x \in \mathcal{L}\}$
  - is a proper tiling of the Euclidean space $\mathbb{R}^3$ where
    
    $\mathcal{L} = \{u_1 \psi_\alpha(1 - \beta) + u_2 \psi_\alpha(1 - \beta^2) + u_3 \psi_\alpha(1 - \beta^3) : u_1, u_2, u_3 \in \mathbb{Z}\}$.

- $\dim_H(\partial F_\alpha) \leq \dim_B(\partial F_\alpha) \approx 2.815$. 

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Rauzy fractals

Let

- $\mathcal{A} = \{1, \ldots, m\}$ be a finite set (alphabet);
- $\mathcal{A}^*$ be the set of finite words over $\mathcal{A}$;
- $\epsilon \in \mathcal{A}$ be the empty word;
- $|W|_a$ be the number of occurrences of the $a \in \mathcal{A}$ in the word $W \in \mathcal{A}^*$;
- $\ell : \mathcal{A}^* \rightarrow \mathbb{R}^m$ be the Abelianisation map

$$\ell(W) = (|W|_1, |W|_2, \ldots, |W|_m)^T.$$ 

A substitution over $\mathcal{A}$ is a non-erasing morphism $\zeta : \mathcal{A}^* \leftrightarrow \mathcal{A}^*$. Define the incidence matrix

$M_\zeta := (\ell(\zeta(1)), \ell(\zeta(2)), \ldots, \ell(\zeta(m))) \in \mathbb{R}^{m \times m}$ and observe that for each $W \in \mathcal{A}^*$ we have $\ell(\zeta(W)) = M_\zeta \cdot \ell(W)$. 

Rauzy fractals

We require $\zeta$ to be

- primitive: there exists a positive integer $n$ such that $M_\zeta^n$ is strictly positive;
- unimodular: $\det(M_\zeta) = \pm 1$;
- Pisot: the Perron-Frobenius eigenvalue if $M_\zeta$ is a Pisot number.

Define

- $K_e$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvector of $M_\zeta$ with respect to the dominant eigenvalue;
- $K_c$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvectors of $M_\zeta$ with respect to the Galois conjugates different from the dominant eigenvalue;
- $K_s$: the subspace of $\mathbb{R}^m$ spanned by the right eigenvectors of $M_\zeta$ with respect to the remaining eigenvalues (if they exist);

Observe:

$$\mathbb{R}^m = K_e \oplus K_c \oplus K_s$$

- $\pi: \mathbb{R}^m \longrightarrow K_c$: a projection parallel to $K_e$ and $K_s$;
- $h: K_c \longrightarrow K_c$: the restriction of the action of $M_\zeta$ on $K_c$, i.e. for all $\mathbf{x} \in \mathbb{R}^m$ we have

$$\pi(M_\zeta \mathbf{x}) = h \circ \pi(\mathbf{x})$$
Rauzy fractals

There exists a uniquely determined invariant set list \( \{ \mathcal{R}_\zeta(a) : a \in A \} \) of compact sets in \( \mathcal{K}_c \) that satisfies the system of set equations

\[
\mathcal{R}_\zeta(a) = \bigcup_{\sigma(b) = PaS} \left( \pi \circ \ell(P) + h(\mathcal{R}_\zeta(a)) \right) \text{ for } a = 1, \ldots, m.
\]

The Rauzy fractal \( \mathcal{R}_\zeta \) associated with \( \zeta \) is the union

\[
\mathcal{R}_\zeta = \bigcup_{a \in A} \mathcal{R}_\zeta(a).
\]
An concrete example

- $A = \{1, 2, 3\}$;
- $\zeta : 1 \mapsto 21, 2 \mapsto 3, 3 \mapsto 31$
- $M_\zeta = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.
- $\chi_{M_\zeta}(x) = 1 - x + 2x^2 - x^3$
- 3 eigenvalues:
  - $\lambda_0 \approx 1.75488 (= \alpha_1)$ (PF)
  - $\lambda \approx 0.122561 + 0.744862i$
  - $\bar{\lambda} \approx 0.122561 - 0.744862i$
- $\zeta$ is irreducible since the algebraic degree of $\lambda$ equals $m$;
- $K_c \cong \mathbb{R}, K_e \cong \mathbb{C}, K_s$ is the trivial space, $h : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto \lambda \cdot z$. 
An concrete example

\[ \zeta : 1 \mapsto 21, \; 2 \mapsto 3, \; 3 \mapsto 31 \]

\[
\begin{align*}
\mathcal{R}_\zeta(1) &= (\lambda + \lambda \cdot \mathcal{R}_\zeta(1)) \cup (\lambda^2 + \lambda \cdot \mathcal{R}_\zeta(3)) \\
\mathcal{R}_\zeta(2) &= \lambda \cdot \mathcal{R}_\zeta(1) \\
\mathcal{R}_\zeta(3) &= \lambda \cdot \mathcal{R}_\zeta(2) U \lambda \cdot \mathcal{R}_\zeta(3)
\end{align*}
\]

\[ \mathcal{R}_\zeta = \mathcal{R}_\zeta(1) U \mathcal{R}_\zeta(2) U \mathcal{R}_\zeta(3) \]

Theorem (Scheicher-Sirvent-S.)

\[ \mathcal{R}_\zeta \text{ and } \mathcal{F}_{\alpha_1} \text{ coincide (up to a linear isomorphism)}. \]
Another example

- $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$;
- $\theta : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 43, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 61$;
- $\chi_{M_\theta}(x) = (x^3 - x^2 - 1)(x^3 - x^2 + 1)$
- The Perron-Frobenius eigenvalue of $M_\theta$ is $\lambda_0 \approx 1.46557 (= \alpha_3)$;
- $\zeta$ is reducible since the algebraic degree of $\lambda$ is smaller than $m$;
- $K_c \cong \mathbb{R}, K_e \cong \mathbb{C}, K_s \cong \mathbb{R}^3$.

**Theorem (Scheicher-Sirvent-S.)**

$\mathcal{R}_\theta(4) \cup \mathcal{R}_\theta(5) \cup \mathcal{R}_\theta(6)$ and $\mathcal{F}_{\alpha_3}$ coincide (up to a linear isomorphism).
Thank you for your presence!

Thank you for your attention!

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