Strong limit theorems for infinite measure-preserving dynamical systems with applications to non-standard continued fractions

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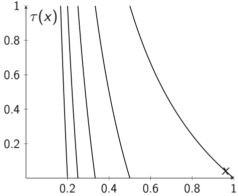
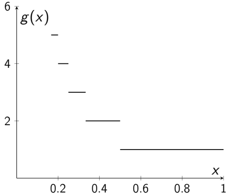
Numeration conference, Liége, May 2023

Each $x \in [0,1]$ can be written as a regular continued fraction given by

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}.$$

We have $a_n(x) = g \circ \tau^{n-1}$,
where $\tau(x) = \{1/x\} = 1/x - \lfloor 1/x \rfloor$ and $g(x) = \lfloor 1/x \rfloor$.





τ has an ergodic invariant probability measure $\mathfrak{m}$ which is equivalent to the Lebesgue measure λ.
We have $\int g \mathrm{d}\mathfrak{m} = \int g \mathrm{d}\lambda = \infty$.

- By Aaronson's theorem we can not get a strong law of large numbers for (a_n).
- Aaronson's theorem says the following:
  Let $(X, \mathcal{B}, \mu, T)$ be an ergodic, probability measure preserving dynamical system, let $f : X \to \mathbb{R}$ such that $\int |f| \mathrm{d}\mu = \infty$ and define $S_n f := \sum_{k=1}^{n} f \circ T^{k-1}$, then we have for any positive valued sequence (d_n) and a.e. $x \in X$ that

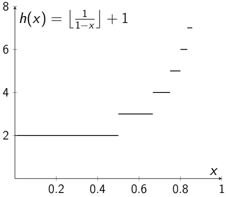
$$\limsup_{n\to\infty} \frac{|S_n f(x)|}{d_n} = \infty \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{|S_n f(x)|}{d_n} = 0. \qquad (1)$$

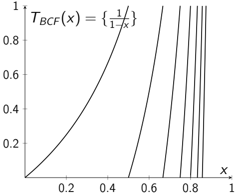
- Since $\int g \mathrm{d}\mathfrak{m} = \infty$, we are in the situation of (1).
- However, by [?] we obtain for a.e. $x \in X$ that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k(x) - \max_{1\le \ell \le n} a_\ell(x)}{n \log n} = \frac{1}{\log 2}.$$

Let's look at a related continued fraction expansion:

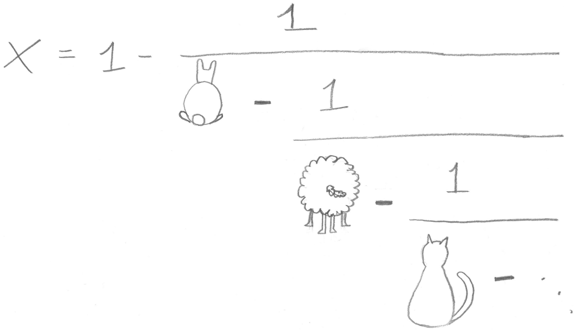
- Each $x \in (0,1)$ can be written as
$$x = 1 - \cfrac{1}{c_1(x) - \cfrac{1}{c_2(x) - \cfrac{1}{c_3(x) - \ddots}}}.$$
- This continued fraction expansion is called backward or Rényi type continued fraction, see [**?**].
- We have $c_n(x) = (h \circ T_{BCF}^{n-1})(x)$,
where $h(x) = \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ and $T_{BCF} = \{\frac{1}{1-x}\}$ with $\{x\} = x - \lfloor x \rfloor$.



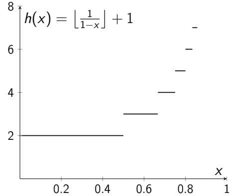


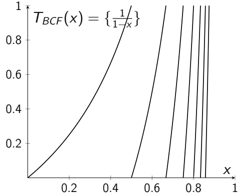
Can we say something about $\sum_{k=1}^{n} c_k(x)$?

Comparing the two continued fraction transformations with each other:
Why are the second ones called “backward” continued fractions?



The backward continued fractions:





We are in a particular situation, we have

- $\mu([0,1]) = \infty$,
- $\int h \, \mathrm{d}\mu = \infty$,
- let $E = [1/2, 1]$, then $\mu(E) = 1$, but still $\int_E h \, \mathrm{d}\mu = \infty$.

Can we say something in general in this situation?

- Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving dynamical system and let $f \notin \mathcal{L}^1(\mu)$.
  Then there are a number of results using trimming, i.e. removing the (or a number of) maximal entries to obtain a strong law, see e.g. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] for results in the dynamicals systems context.

- Let $(X, \mathcal{B}, \mu, T)$ be a conservative, infinite σ-finite measure preserving dynamical system and let $f \in \mathcal{L}^1(\mu)$.
Then by the second part of Aaronson's theorem we obtain for any positive valued sequence (d_n) and a.e. $x \in X$ that

$$\limsup_{n\to\infty} \frac{|S_n f(x)|}{d_n} = \infty \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{|S_n f(x)|}{d_n} = 0.$$

- Adding additional summands can help:
[?] gives conditions on the system $(X, \mathcal{B}, \mu, T)$ with μ infinite such that there exists a sequence of positive numbers (d_n) such that for all non-negative $f \in \mathcal{L}^1(\mu)$ and a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{S_{N+m(N,x)} f(x)}{d_N} = \int f \mathrm{d}\mu.$$

(We will have a look at the precise definition of m in the following.)

An example of the last statement:

- Let $T : [0,1] \to [0,1]$ be the Farey map

$$T(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in [0,1/2] \\ \frac{1-x}{x} & \text{if } x \in [1/2,1]. \end{cases}$$

- It preserves the infinite invariant measure $\mathrm{d}\mu(x) = \frac{1}{x \log 2}\mathrm{d}x$.
- Setting $E = [1/2,1]$, then $\mu(E) = 1$, we denote by φ_E the *first return time*

$$\varphi_E : E \to \mathbb{N}, \qquad \varphi_E(x) := \min\left\{k \geq 1 \,:\, T^k(x) \in E\right\}.$$

- The first return time is finite for μ-a.e. $x \in E$, so we define the *induced map*

$$T_E : E \to E \quad \text{by} \quad T_E(x) := T^{\varphi_E(x)}(x)$$

(an ergodic measure-preserving transformation of the probability space $(E, \mathcal{B}|_E, \mu)$).
- $\varphi_E \circ T_E^{k-1}(x)$ gives the kth continued fraction entry of x.

- We denote the *longest excursion out of E beginning in the first N-steps* (defined for μ-a.e. $x \in X$) by

$$m(N, E, x) := 1 + \max\left\{k \geq 1 \,:\, \exists \ell \in \{1, \dots, N+1\} \right.$$
$$\left. \text{s.t. } T^{\ell+j}(x) \notin E, \forall j = 0, \dots, k-1 \right\}.$$

- Let's look at an example: in red: $k : T^k(x) \in E$

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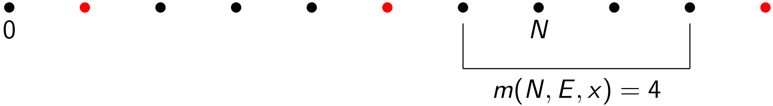

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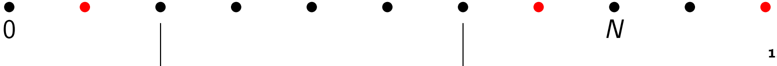

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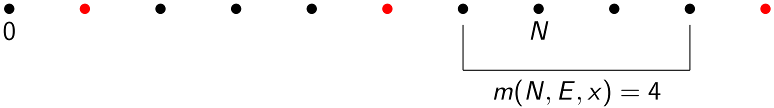

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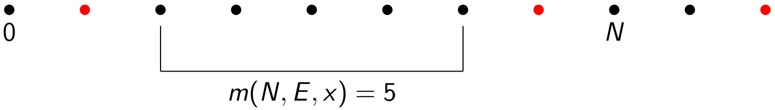

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For the Farey map T and $f \in \mathcal{L}^1(\mu)$, we have for μ-a.e. $x \in X$

$$\lim_{N\to\infty} \frac{S_{N+m(N,E,x)} f \;\log N}{N \log 2} = \int f \,\mathrm{d}\mu.$$

Similar statements hold for other infinite measure preserving maps.

Back to the literature review:

- If $(X, \mathcal{B}, \mu, T)$ is a conservative, infinite σ-finite measure preserving dynamical system, and $f \notin \mathcal{L}^1(\mu)$, then a strong law of large numbers might be possible:
  The easiest case would be $f \equiv C$, then for every $x \in X$:
  $$\lim_{N \to \infty} \frac{S_N f(x)}{N} = C.$$
  [?] and [?] give conditions for a strong law of large numbers for more interesting observables than $f \equiv C$ and also other norming sequences than $(d_N) = (N)$.
- However, in all cases cosidering an infinite measure, we always assume that $\int_E |f| \, \mathrm{d}\mu < \infty$ if $\mu(E) < \infty$.
- Aim for today: Let's put some light on the case $\mu(X) = \infty$ and there exists E with $\mu(E) < \infty$ and $\int_E f \, \mathrm{d}\mu = \infty$.

Remember the backward continued fraction transformation:

$$x = 1 - \cfrac{1}{c_1(x) - \cfrac{1}{c_2(x) - \ddots}}.$$

We have for a.e. $x \in [0,1]$ that

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{j=1}^{N+m(N,E,x)} c_j(x) - \max\left\{ 2\, m(N,E,x), \max_{1\le k\le N+m(N,E,x)} c_k(x) \right\} \right) = 3,$$

see [?].

Another continued fraction expansion: Even continued fractions:

$$x = \cfrac{1}{2h_1(x) + \cfrac{\epsilon_1}{2h_2 + \cfrac{\epsilon_2}{2h_3(x) + \ddots}}}, \tag{2}$$

where $h_j \in \mathbb{N}$ and $\epsilon_j \in \{-1, 1\}$.
We have for a.e. $x \in [0, 1]$

$$\lim_{N\to\infty} \frac{1}{N} \left[\sum_{j=1}^{N+m(N,E,x)} 2h_j(x) - \max\left\{ m(N,E,x), \max_{1\leq k\leq N+m(N,E,x)} 2h_k(x) \right\} \right] = 3,$$

see [?].
A similar statement could be made for the odd-odd continued fraction expansion.

Further (related) statements: Let (c_n) again be the digits of the backward continued fraction expansion.

- Let $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $g(n) = o(n)$ and $g(1) = K$. Then we have for a.e. $x \in [0, 1]$

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{j=1}^{N} g(c_j(x)) - \max_{1\leq k\leq N} g(c_k(x)) \right) = K.$$

- If additionally $g(n) \lesssim n/(\log\log n)^u$ with $u > 1$, we have for a.e. $x \in [0, 1]$

$$\lim_{N\to\infty} \frac{1}{N} \left(\sum_{j=1}^{N} g(c_j(x)) \right) = K.$$

- If on the other hand $g(n) \gtrsim n$, then one might need to deduce more than only the largest entry and divide by another norming sequence than N.

The main theorem:
Let $(X, \mathcal{B}, \mu, T)$ be an ergodic, conservative, σ-finite system with $\mu(X) = \infty$ and $\mu(E) = 1$.
Let $g : X \to \mathbb{R}_{\geq 0}$ be a measurable observable and let $\mathcal{P}_E$ be the partition of E induced by T_E. We assume that:

(i) We assume that for any sequence $(\phi_n)_{n\in\mathbb{N}}$ piecewise constant on $\mathcal{P}_E$, the sequence $(\phi_n \circ T_E^{n-1})_{n\geq 1}$ is fast enough ψ-mixing.

(ii) Let $A_{>n} := \{x \in E : \varphi_E(x) > n\}$
$\mu(A_{>n})$ does not grow too fast, e.g. $\mu(A_{>n}) = 1/n$ would work.

(iii) There exists a constant $\kappa > 0$ such that $\mu(g > n) \sim \kappa\mu(A_{>n})$.

(iv) The function g is locally constant on $\mathcal{P}_E$ and $g \notin \mathcal{L}^1(E, \mu)$.

(v) There exists $c \in \mathbb{R}$ such that $g \equiv c$ on $X \backslash E$.

Then, for μ-a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{S_{N+m(N,E,x)}g(x) - \max_{1\leq k\leq N+m(N,E,x)}(g \circ T^{k-1})(x) - c\, m(N, E, x)}{N}$$
$$= c + \kappa.$$

Corrollary:
If additionally to the conditions of the previous theorem we have for $M, N \in \mathbb{N}$ that

$$\mu\left(\{g > N\} \cap \{\varphi_E > M\}\right) \asymp \mu\left(g > N\right) \mu\left(\varphi_E > M\right),$$

then for μ-a.e. $x \in X$

$$\lim_{N\to\infty} \frac{S_{N+m(N,E,x)}g(x) - \max\left\{\max_{1\leq k\leq N+m(N,E,x)}(g \circ T^{k-1})(x), c\, m(N,E,x)\right\}}{N}$$
$$= c + \kappa.$$

Outlook:

- Is the current theorem applicable for further continued fraction classes?
- In terms of the infinite measure set, so far we only cover some cases where the return time sets are regularly varying with index 1, a generalization for broader classes of infinity are needed.
- So far the results heavily rely on the ψ-mixing property, weakening this assumption might only be possible if we increase $m(N, E, x)$ (add more digits).
- A special class are α-continued fractions with $\alpha \in (0, 1/2)$. Probably, the ψ-mixing condition does not hold for them.
- It would also be interested to look at random dynamics, e.g. using the regular and the backward cf transformation randomly.

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