The destiny of a waltz and some other stories

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CNRS / Université Paris Cité and Télécom Paris, IPP

Numeration 2023, 14 February 2023, Liège University

A session on the occasion of Christiane Frougny's 75th birthday

Based on the paper that introduced the rational base numeration systems:

- *Powers of rationals modulo 1 and rational base number systems*,
  Israel J. Math., 2008 with Sh. Akiyama and Ch. Frougny

and subsequent more recent works on the subject:

- *Trees and languages with periodic signature*,
  Indagationes Mathematicae 2017 with V. Marsault
- *On subtrees of the representation tree in rational base numeration systems*,
  DMTCS 2018 with Sh. Akiyama and V. Marsault

I

From a boosted odometer to a numeration system

0
1
2
3
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11
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14
15
16

0	0
2	1
1	2
0	3
2	4
1	5
0	6
2	7
1	8
0	9
2	10
1	11
0	12
2	13
1	14
0	15
2	16

0	0
2	1
1	2
0	3
2	4
1	5
0	6
2	7
1	8
0	9
2	10
1	11
0	12
2	13
1	14
0	15
2	16

00	0
02	1
21	2
10	3
12	4
01	5
20	6
22	7
11	8
00	9
02	10
21	11
10	12
12	13
01	14
20	15
22	16

00	0
02	1
21	2
10	3
12	4
01	5
20	6
22	7
11	8
00	9
02	10
21	11
10	12
12	13
01	14
20	15
22	16

000	0
002	1
021	2
210	3
212	4
101	5
120	6
122	7
011	8
200	9
202	10
221	11
110	12
112	13
001	14
020	15
022	16

000	0
002	1
021	2
210	3
212	4
101	5
120	6
122	7
011	8
200	9
202	10
221	11
110	12
112	13
001	14
020	15
022	16

0000	0
0002	1
0021	2
0210	3
0212	4
2101	5
2120	6
2122	7
1011	8
1200	9
1202	10
1221	11
0110	12
0112	13
2001	14
2020	15
2022	16

0000	0
0002	1
0021	2
0210	3
0212	4
2101	5
2120	6
2122	7
1011	8
1200	9
1202	10
1221	11
0110	12
0112	13
2001	14
2020	15
2022	16

00000	0
00002	1
00021	2
00210	3
00212	4
02101	5
02120	6
02122	7
21011	8
21200	9
21202	10
21221	11
10110	12
10112	13
12001	14
12020	15
12022	16

00000	0
00002	1
00021	2
00210	3
00212	4
02101	5
02120	6
02122	7
21011	8
21200	9
21202	10
21221	11
10110	12
10112	13
12001	14
12020	15
12022	16

000000	0
000002	1
000021	2
000210	3
000212	4
002101	5
002120	6
002122	7
021011	8
021200	9
021202	10
021221	11
210110	12
210112	13
212001	14
212020	15
212022	16

The base $\frac{3}{2}$ numeration system — the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

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Modified division algorithm $\quad N \in \mathbb{N}$

$$\begin{aligned} N_0 &= N \\ 2\,N_0 &= 3\,N_1 + a_0 & \qquad a_0 \in A \\ 2\,N_1 &= 3\,N_2 + a_1 & \qquad a_1 \in A \\ &\dots \end{aligned}$$

$$N = \sum_0^k a_i \, \frac{1}{2} \left(\frac{3}{2}\right)^i \qquad\qquad \langle N \rangle_{\frac{3}{2}} = \; a_k \, a_{k-1} \dots a_1 \, a_0$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

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Modified division algorithm $\quad 5 \in \mathbb{N}$

$$N_0 = 5$$

$$2\,N_0 = 2 \cdot 5 = 3 \cdot 3 + 1 \qquad 1 \in A$$

$$2\,N_1 = 2 \cdot 3 = 3 \cdot 2 + 0 \qquad 0 \in A$$

$$2\,N_2 = 2 \cdot 2 = 3 \cdot 1 + 1 \qquad 1 \in A$$

$$2\,N_3 = 2 \cdot 1 = 3 \cdot 0 + 2 \qquad 2 \in A$$

$$5 = \frac{1}{2} \left[\left(\left((2) \cdot \frac{3}{2} + 1 \right) \cdot \frac{3}{2} + 0 \right) \cdot \frac{3}{2} + 1 \right] \qquad \langle 5 \rangle_{\frac{3}{2}} = 2\,1\,0\,1$$

The base $\frac{3}{2}$ numeration system — the Euclidean approach

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Theorem

Every N *in* $\mathbb{N}$ *has an integer representation in the* $\frac{3}{2}$*-system.*

It is the unique finite $\frac{3}{2}$*-representation of* N *.*

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We call this representation the $\frac{3}{2}$-expansion of N

and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

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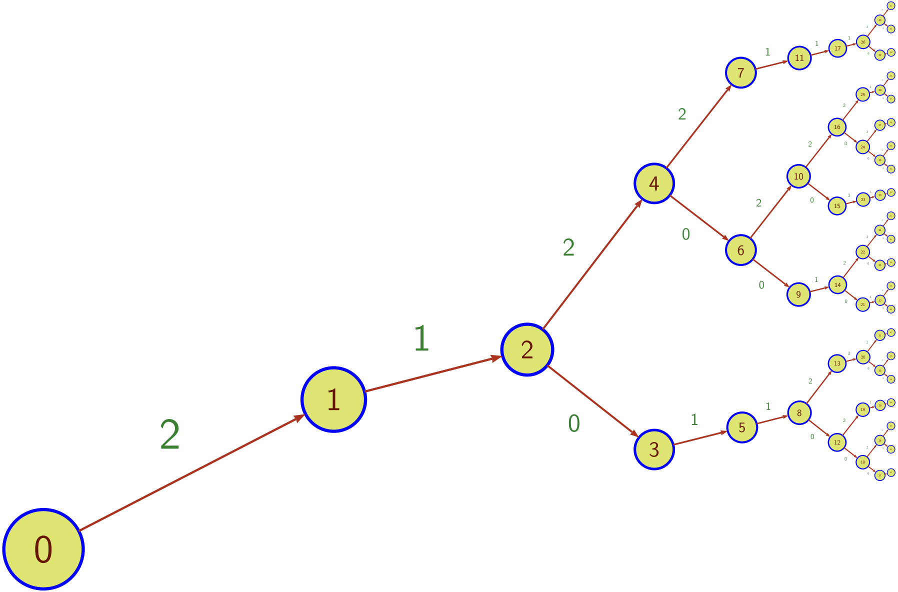
We call this representation the $\frac{3}{2}$-expansion of N
and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

$$L_{\frac{3}{2}} = \left\{ \langle N \rangle_{\frac{3}{2}} \,\middle|\, N \in \mathbb{N} \right\} = ????$$

	0	212211	17
2	1	2101100	18
21	2	2101102	19
210	3	2101121	20
212	4	2120010	21
2101	5	2120012	22
2120	6	2120201	23
2122	7	2120220	24
21011	8	2120222	25
21200	9	2122111	26
21202	10	21011000	27
21221	11	21011002	28
210110	12	21011021	29
210112	13	21011210	30
212001	14	21011212	31
212020	15	21200101	32
212022	16	21200120	33



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The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$L_{\frac{3}{2}}$ prefix-closed $\Longrightarrow$ $L_{\frac{3}{2}}$ spans the edges of a subtree $T_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $T_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented by the label of the path from the root to that node.

These labels give *the radix order* in $L_{\frac{3}{2}}$.

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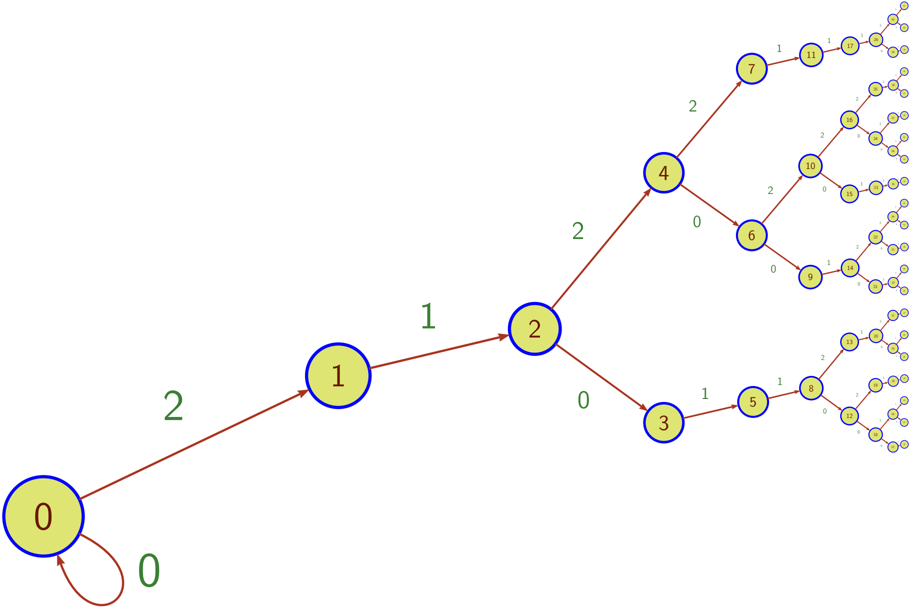
These labels give *the radix order* in $L_{\frac{3}{2}}$.

Any two distinct subtrees of $T_{\frac{3}{2}}$ are not isomorphic.

II

Periodic signatures



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Meta theorem

The $T_{\frac{p}{q}}$ are characterised by their *periodic signature*.

Signature of a tree

Definition

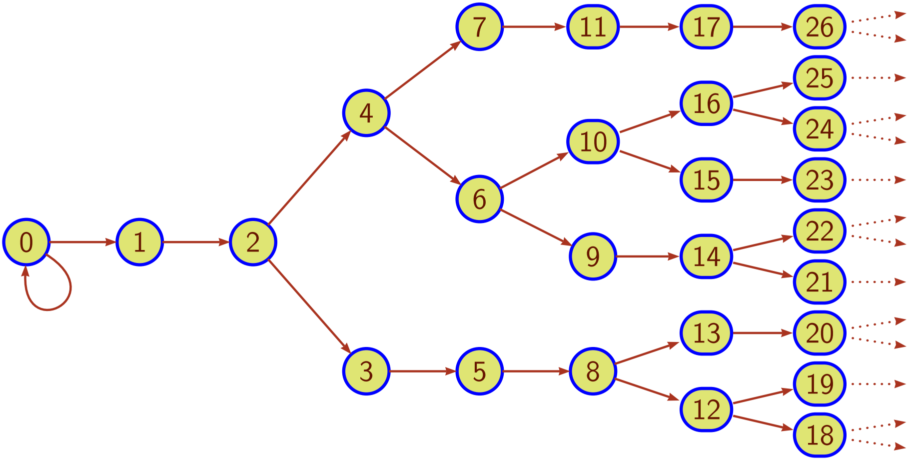
Signature of an ordered tree $\mathcal{T}$ =

sequence of the degrees of the nodes

in the breadth-first traversal of $\mathcal{T}$

Signature of a tree

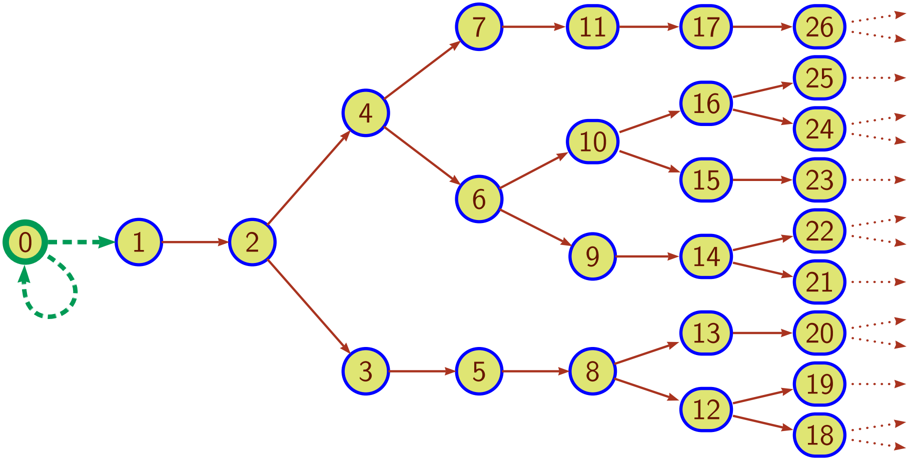
Signature = sequence of the degrees



$\mathbf{s} =$

Signature of a tree

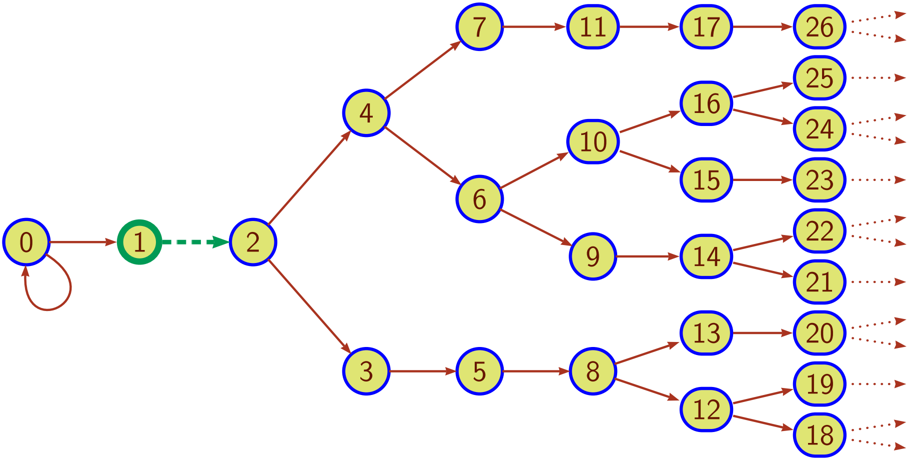
Signature = sequence of the degrees



$\mathbf{s} = 2$

Signature of a tree

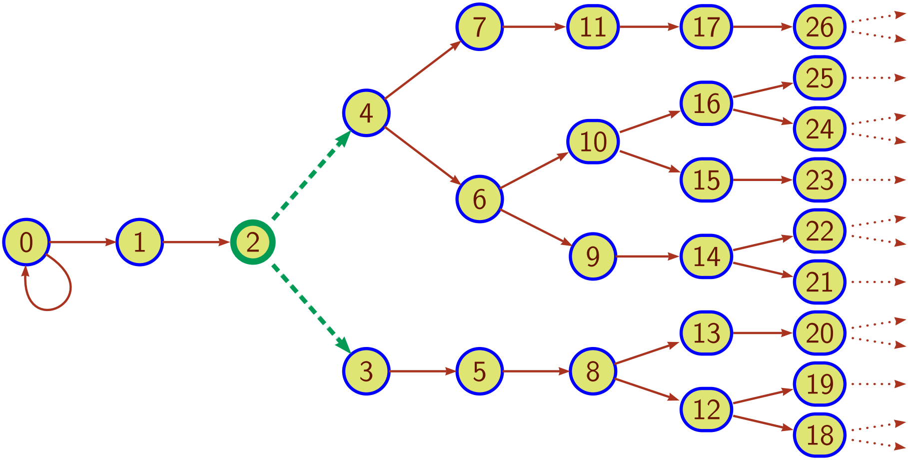
Signature = sequence of the degrees



s = 2 1

Signature of a tree

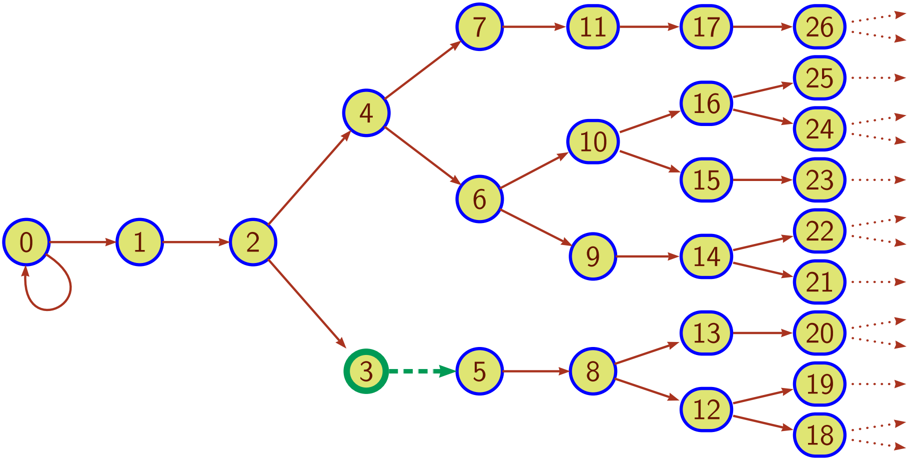
Signature = sequence of the degrees



$\mathbf{s} = \ 2 \ 1 \ 2$

Signature of a tree

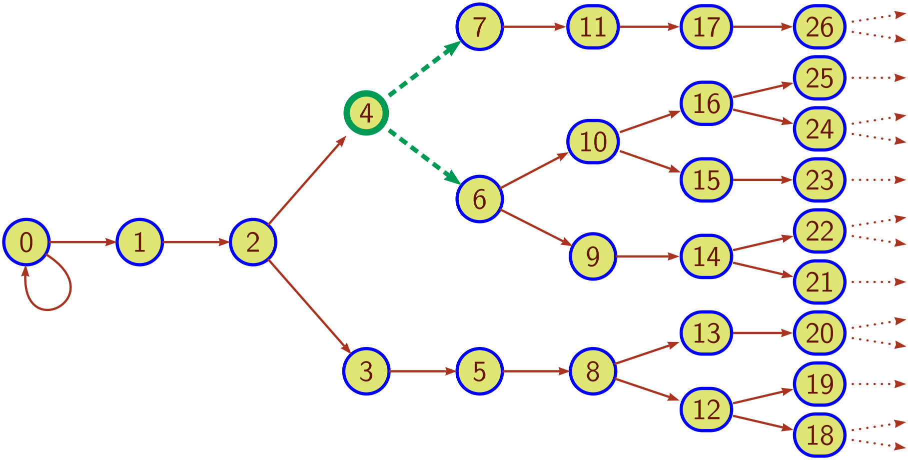
Signature = sequence of the degrees



s = 2 1 2 1

Signature of a tree

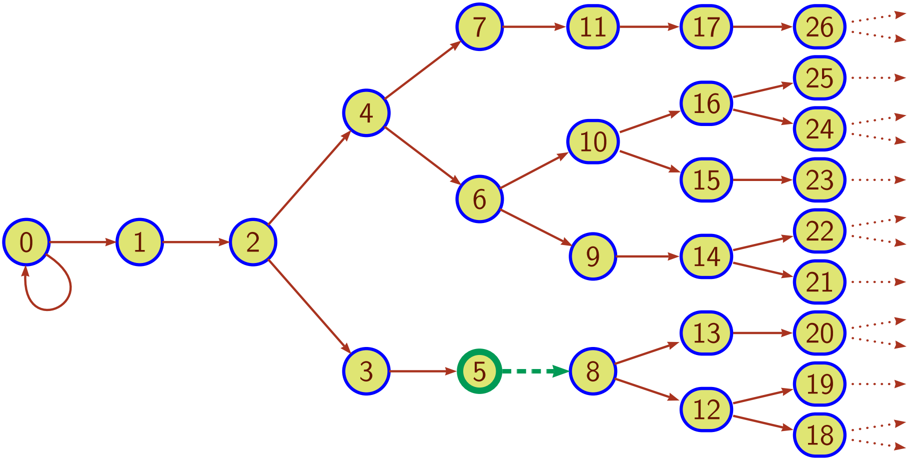
Signature = sequence of the degrees



$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2$

Signature of a tree

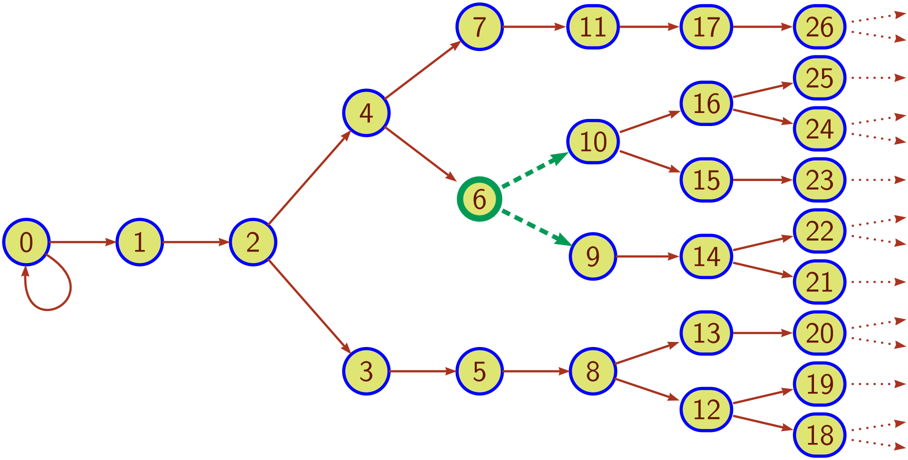
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

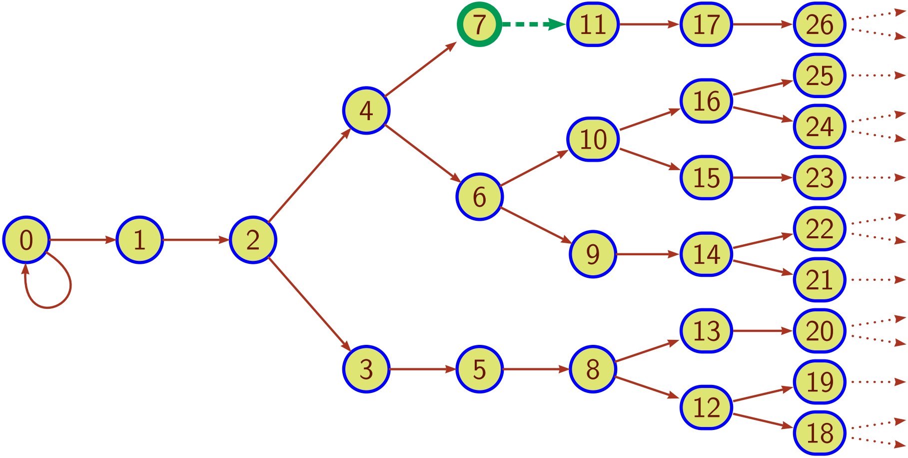
Signature = sequence of the degrees



s = 2 1 2 1 2 1 2

Signature of a tree

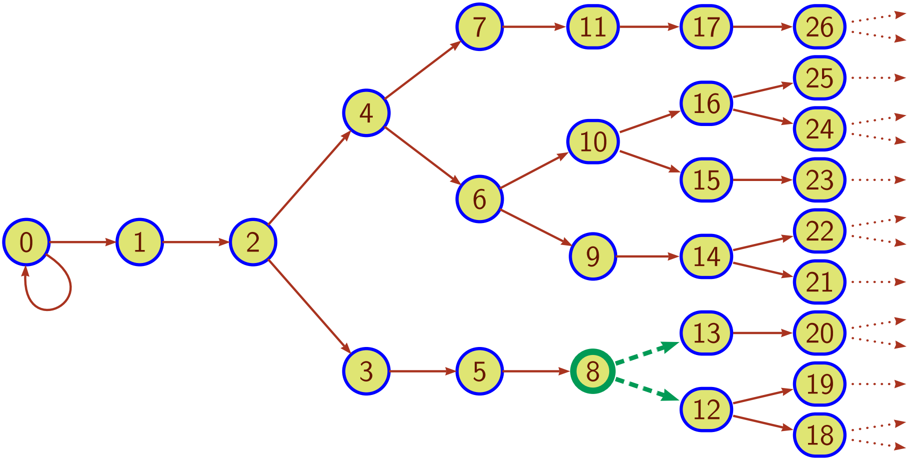
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

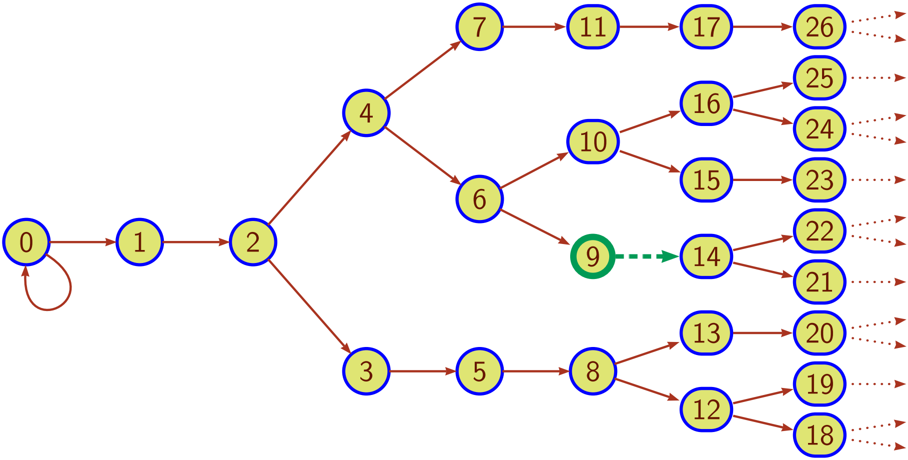
Signature = sequence of the degrees



$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2$

Signature of a tree

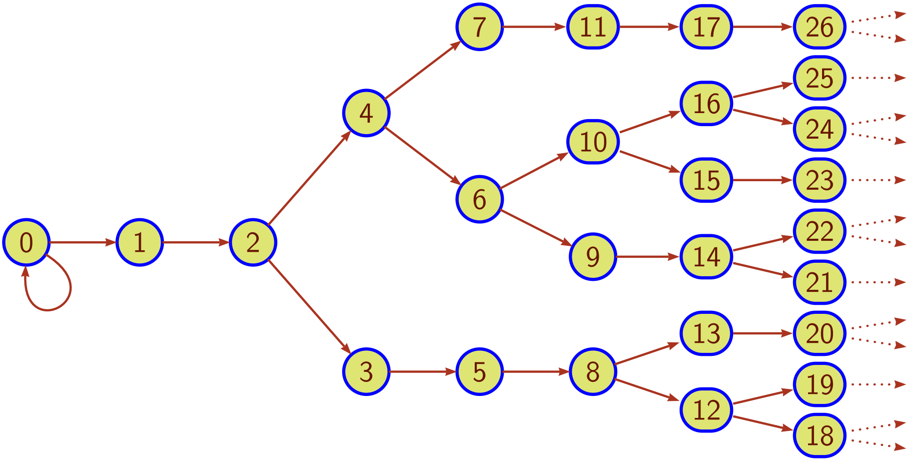
Signature = sequence of the degrees



$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1$

Signature of a tree

Signature = sequence of the degrees



$$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

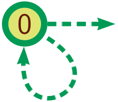
Tree from a signature

Signature = sequence of the degrees

$$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

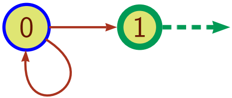
Signature = sequence of the degrees



$\mathbf{s} = \; {\color{red}2} \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$

Tree from a signature

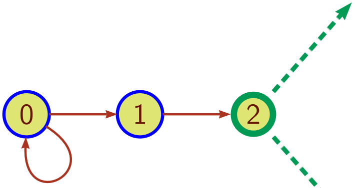
Signature = sequence of the degrees



$\mathbf{s} =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 $\cdots$

Tree from a signature

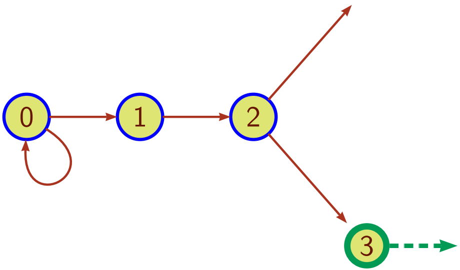
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

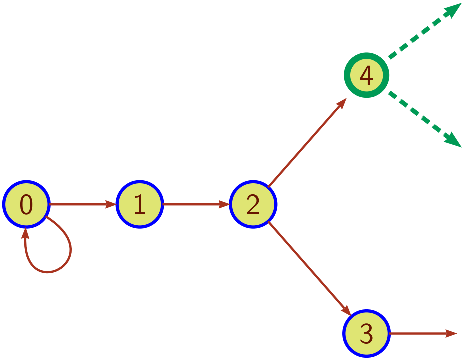
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

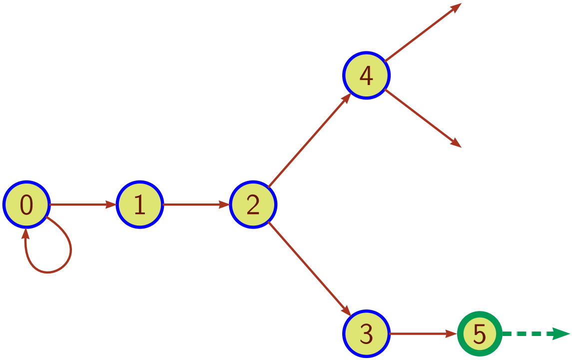
Signature = sequence of the degrees



$$\mathbf{s} = \; 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

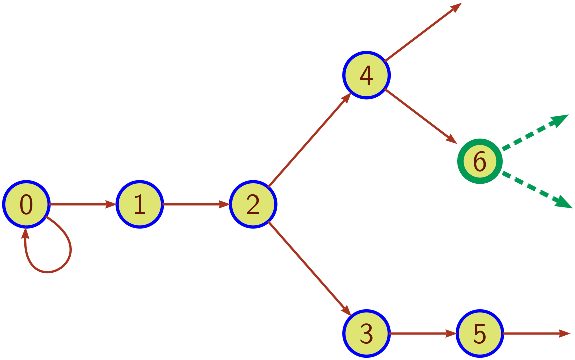
Signature = sequence of the degrees



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

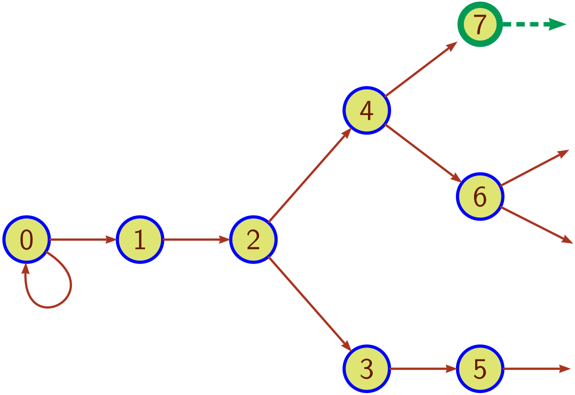
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ {\color{red}2}\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

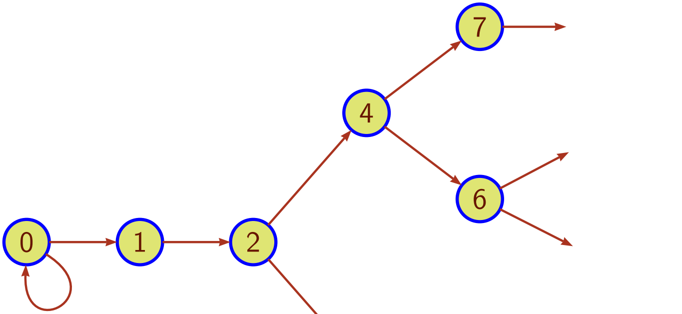
Signature = sequence of the degrees



$\mathbf{s} =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 $\cdots$

Tree from a signature

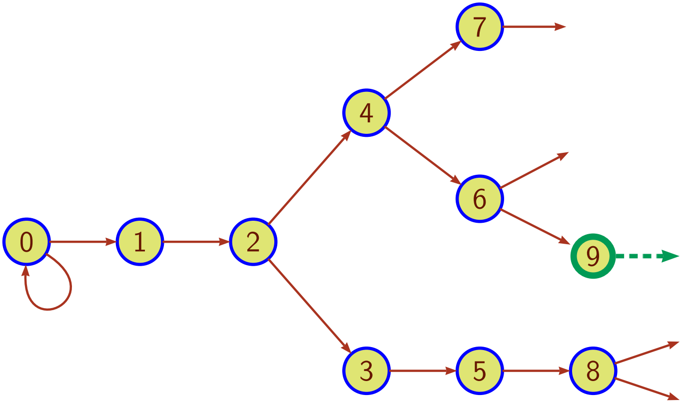
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ {\color{red}2}\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

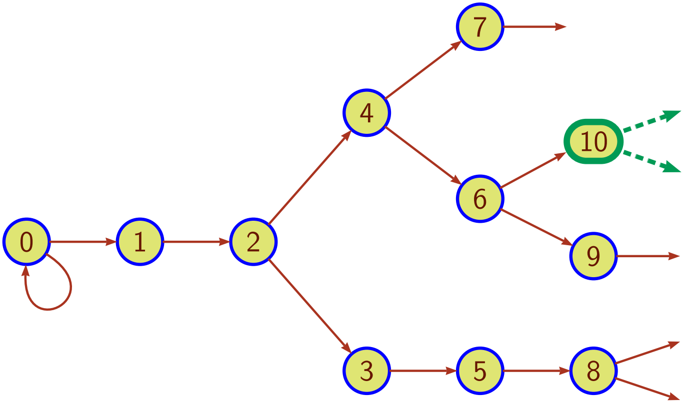
Signature = sequence of the degrees



$\mathbf{s} =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 $\cdots$

Tree from a signature

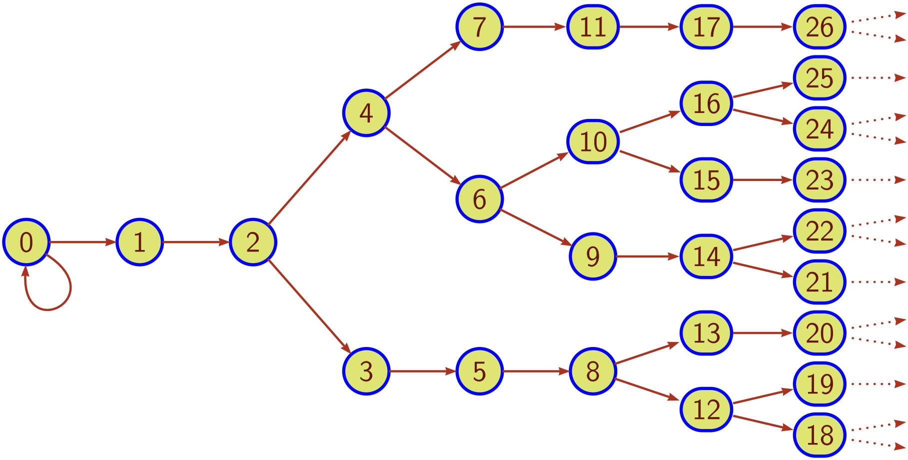
Signature = sequence of the degrees



$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ {\color{red}2}\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

Signature = sequence of the degrees



$$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

Labelled signature of a labelled tree

Arcs of $\mathcal{T}$ labelled in an ordered alphabet A

Definition

Labelled signature of an ordered tree $\mathcal{T}$ =

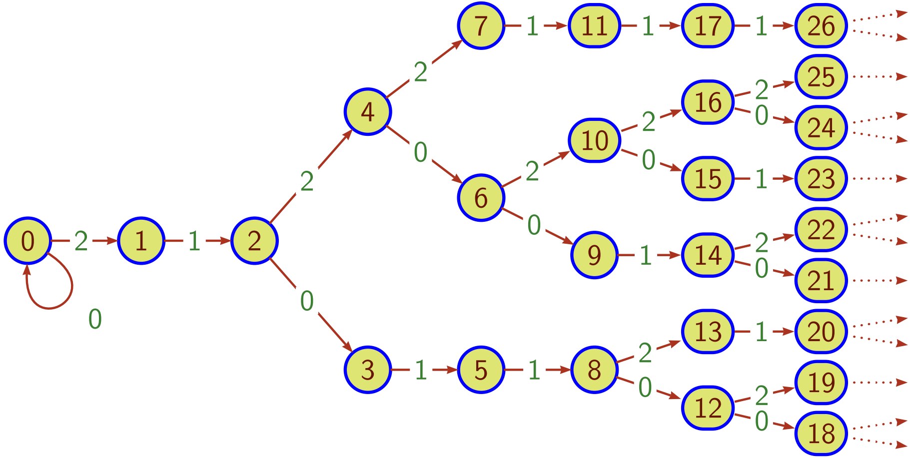
signature of $\mathcal{T}$ +

sequence of the labels of the arcs

in the breadth-first traversal of $\mathcal{T}$

labelled signature $(\mathbf{s}, \boldsymbol{\lambda})$

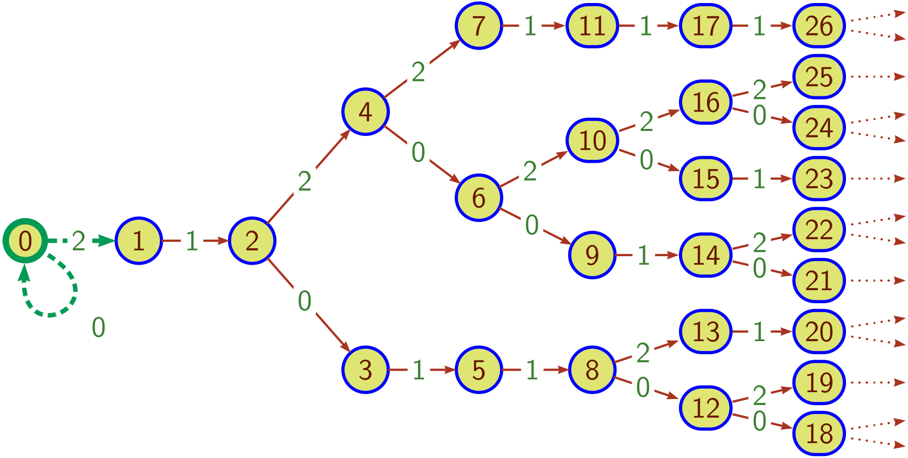
Labelled signature of a labelled tree



$\mathbf{s} =$

$\boldsymbol{\lambda} =$

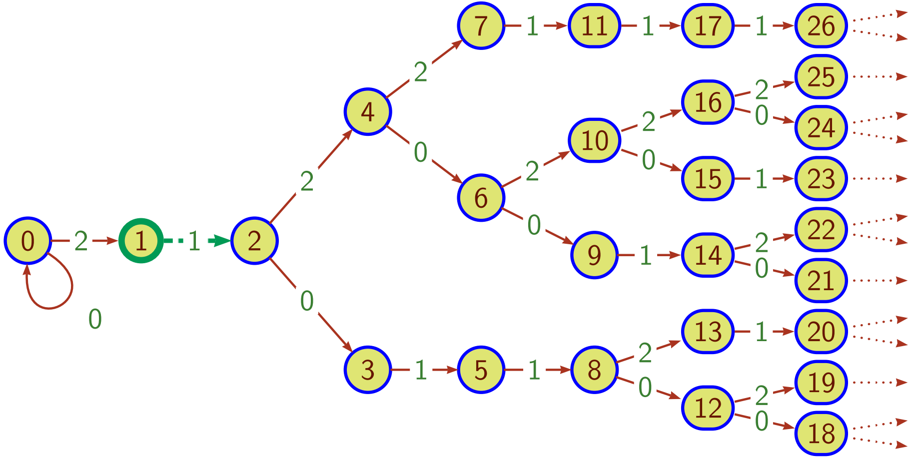
Labelled signature of a labelled tree



$\mathbf{s} = \ 2$

$\boldsymbol{\lambda} = 0\,2$

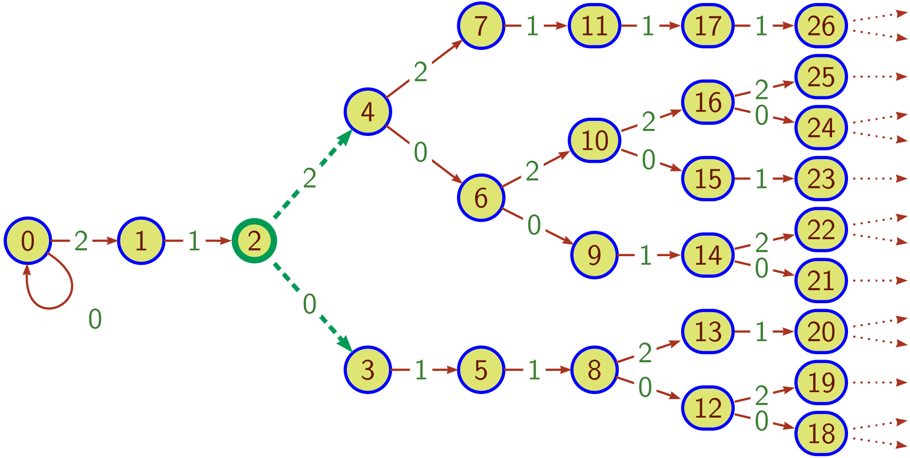
Labelled signature of a labelled tree



$\mathbf{s} = \ 2 \ 1$

$\boldsymbol{\lambda} = 0\,2\,1$

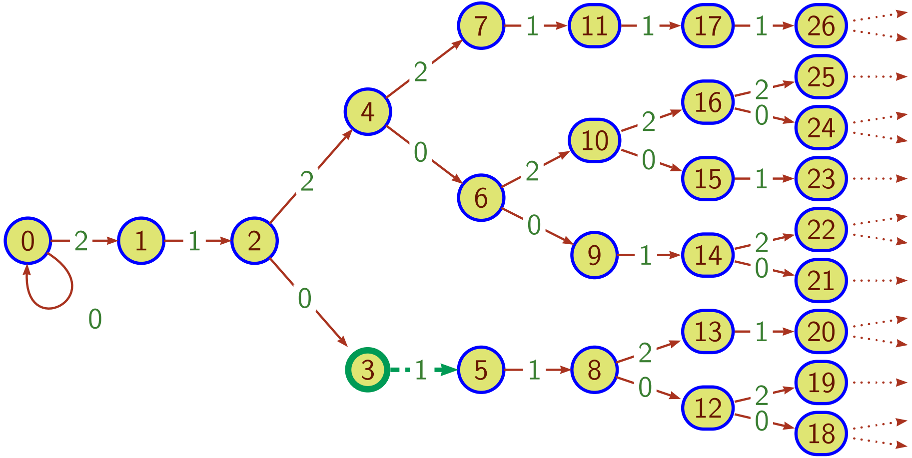
Labelled signature of a labelled tree



$\mathbf{s} = \ 2\ 1\ 2$

$\boldsymbol{\lambda} = 0\,2\,1\,0\,2$

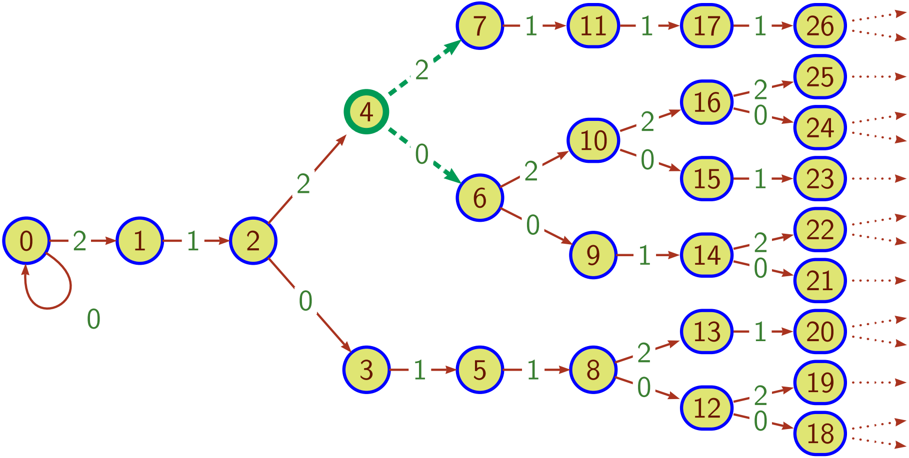
Labelled signature of a labelled tree



$$\mathbf{s} = \ 2\ 1\ 2\ 1$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1$$

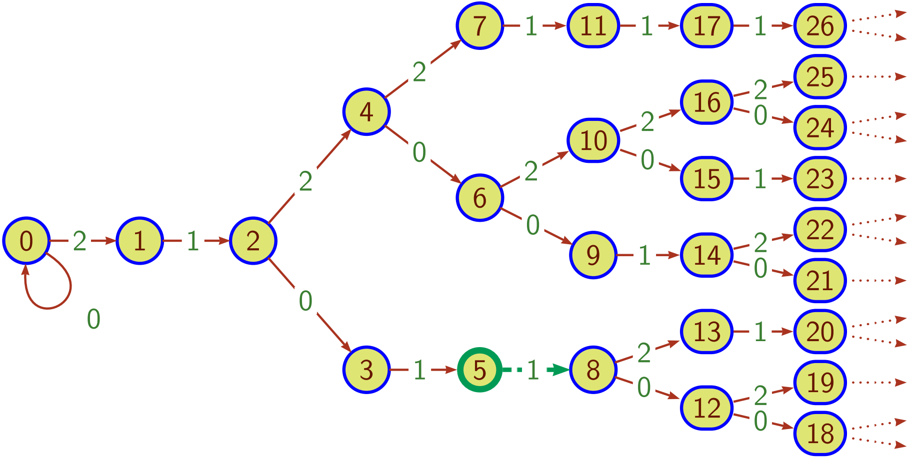
Labelled signature of a labelled tree



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2$$

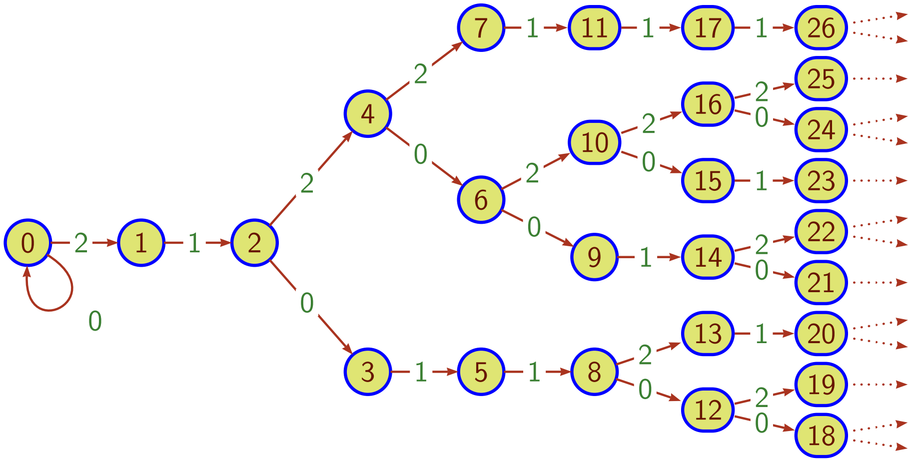
Labelled signature of a labelled tree



$$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ 1$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1$$

Labelled signature of a labelled tree



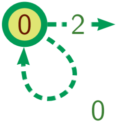
$$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\;\cdots$$

Labelled tree from a labelled signature

$$\mathbf{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$
$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,\cdots$$

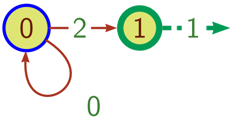
Labelled tree from a labelled signature



$\mathbf{s} = \color{red}{2}\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

$\boldsymbol{\lambda} = \color{red}{0\,2}\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\,0\,2\ 1\ \cdots$

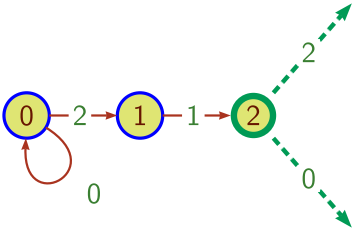
Labelled tree from a labelled signature



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\ \cdots$$

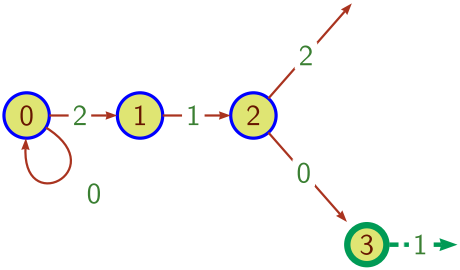
Labelled tree from a labelled signature



$\mathbf{s} = \; 2 \;\; 1 \;\; {\color{red}2} \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \cdots$

$\boldsymbol{\lambda} = 0\,2\,1\,{\color{red}0\,2}\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1 \cdots$

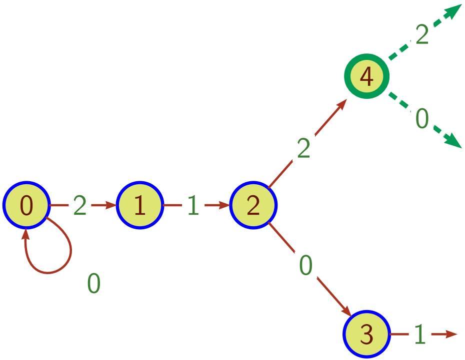
Labelled tree from a labelled signature



$$\mathbf{s} = \ 2\ 1\ 2\ \color{red}{1}\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,\color{red}{1}\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\ \cdots$$

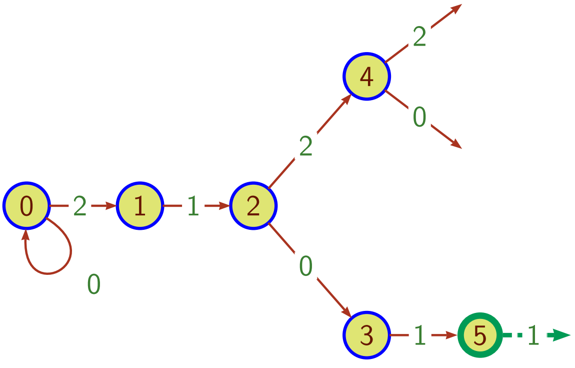
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; {\color{red}2} \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\;1\;0\,2\;1\;{\color{red}0\,2}\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;0\,2\;1\;\cdots$$

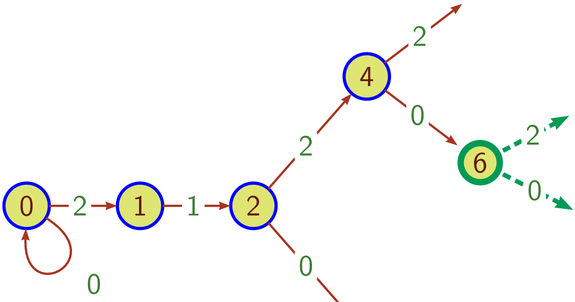
Labelled tree from a labelled signature



$$\mathbf{s} = \ 2\ 1\ 2\ 1\ 2\ {\color{red}1}\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,{\color{red}1}\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\ \cdots$$

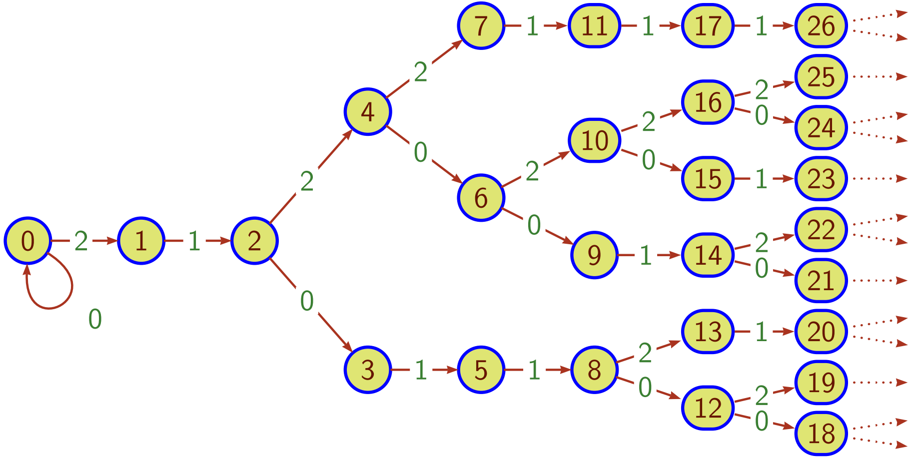
Labelled tree from a labelled signature



$$\mathbf{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; {\color{red}2} \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; {\color{red}0\,2} \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; 0\,2 \; 1 \; \cdots$$

Labelled tree from a labelled signature



$$\mathbf{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ \cdots$$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Theorem

The (labelled) signature of $T_{\frac{p}{q}}$ is purely periodic.

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\mathbf{s} \ = \ \mathbf{r}^{\omega}$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\mathbf{s} \;=\; \mathbf{r}^{\omega}$$

Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} \;=\; (r_0, r_1, \ldots, r_{q-1}) \qquad\qquad \sum_{i=0}^{q-1} r_i \;=\; p$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

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Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1}) \qquad \sum_{i=0}^{q-1} r_i = p$$

Example

Rhythms of dir. par. $(3, 5)$: $(3, 1, 1)$ $(2, 2, 1)$ $(1, 2, 2)$

Rhythm

p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\texttt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$

Rhythm

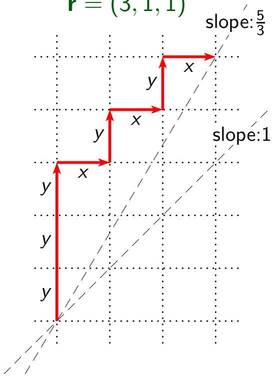
p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\mathtt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$

$\mathbf{r} = (3, 1, 1)$



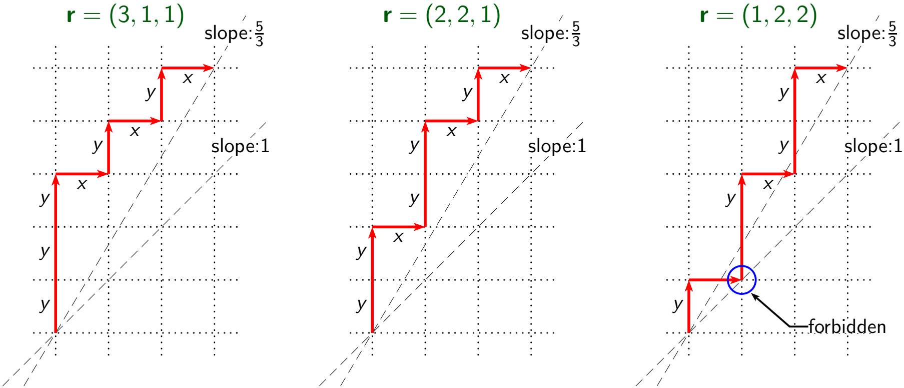
Rhythm

p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\mathtt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$



Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\mathtt{path}(\mathbf{r})$ Christoffel word

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$\mathbf{r}$ Christoffel rhythm if $\mathtt{path}(\mathbf{r})$ Christoffel word

$\mathtt{path}(\mathbf{r})$ Christoffel word if no integer point between $\mathtt{path}(\mathbf{r})$ and slope

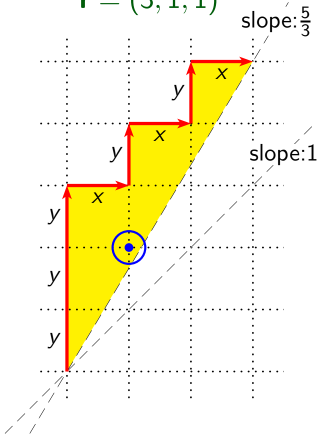
Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

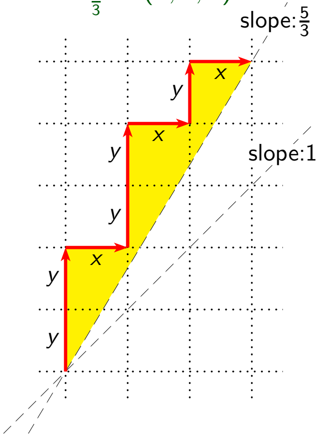
p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\texttt{path}(\mathbf{r})$ Christoffel word

$\texttt{path}(\mathbf{r})$ Christoffel word if no integer point between $\texttt{path}(\mathbf{r})$ and slope

$\mathbf{r} = (3, 1, 1)$ $\qquad$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$





Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The signature of $T_{\frac{p}{q}}$ *is purely periodic of period* $\mathbf{r}_{\frac{p}{q}}$.

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\quad$ A *ordered* alphabet

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\qquad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) = (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

r rhythm of dir. par. (q, p) $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\gamma_i \in A$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ A *ordered* alphabet

A purely periodic labelled signature

$$(\mathbf{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\gamma_i \in A$

Definition

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$$

$\boldsymbol{\gamma} = u_0\, u_1 \cdots u_{q-1}$ factorisation induced by $\mathbf{r}$ $|u_i| = r_i$

$\boldsymbol{\gamma}$ consistent with $\mathbf{r}$ every u_i increasing word

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} \;=\; \big(\, 0,\, (q \,\%\, p),\, (2\,q \,\%\, p), \ldots,\, ((p-1)\,q \,\%\, p) \,\big) \;\; .$$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0, (q \,\%\, p), (2\,q \,\%\, p), \ldots, ((p-1)\,q \,\%\, p)\big) \;.$$

Proposition

$\gamma_{\frac{p}{q}}$ *is consistent with* $\mathbf{r}_{\frac{p}{q}}$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The labelled signature of $T_{\frac{p}{q}}$ *is purely periodic of period* $(\mathbf{r}_{\frac{p}{q}}, \boldsymbol{\gamma}_{\frac{p}{q}})$.

The tree $T_{\mathbf{r}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ rhythm of directing parameter (q, p) $\qquad$ $\boldsymbol{\gamma}_{\mathbf{r}}$ special labelling

Definition

$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \boldsymbol{\gamma}_{\mathbf{r}}^{\omega})$

Theorem

$L_{\mathbf{r}}$ *is the representation of integers in base* $\frac{p}{q}$
with non-canonical set of digits.

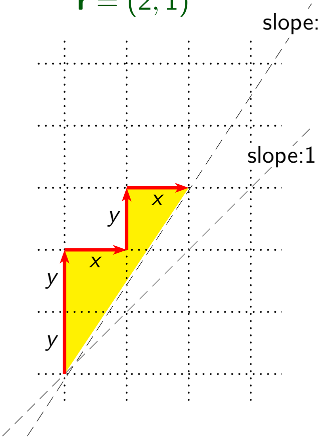
Corollary

$L_{\frac{p}{q}}$ *is the image of* $L_{\mathbf{r}}$ *by*
a finite letter-to-letter sequential right transducer.

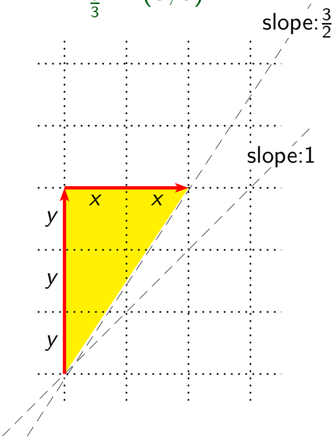
Rhythm of directing parameters $(2,3)$

Two possibilities

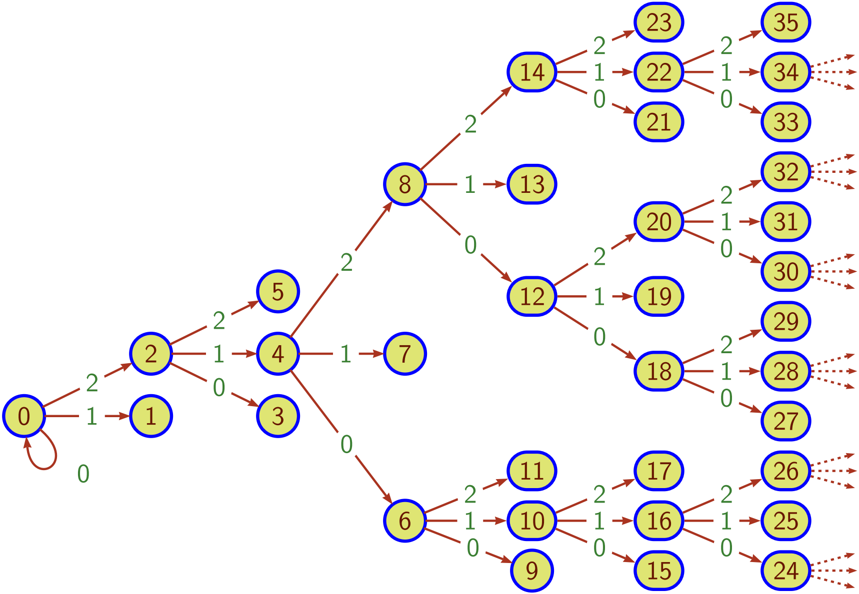
$\mathbf{r} = (2,1)$



$\mathbf{r}_{\frac{5}{3}} = (3,0)$

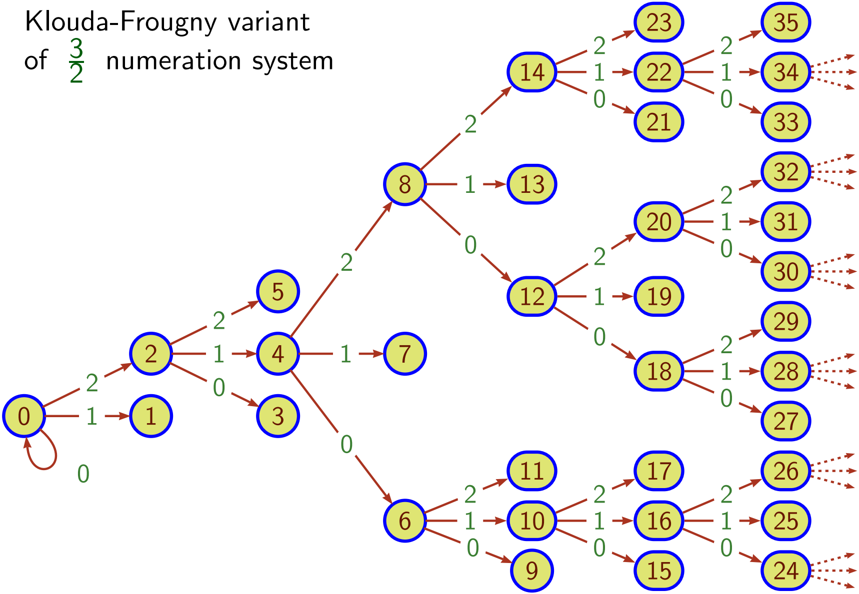


Tree of rhythm $(3, 0)$



Tree of rhythm $(3, 0)$

Klouda-Frougny variant
of $\frac{3}{2}$ numeration system



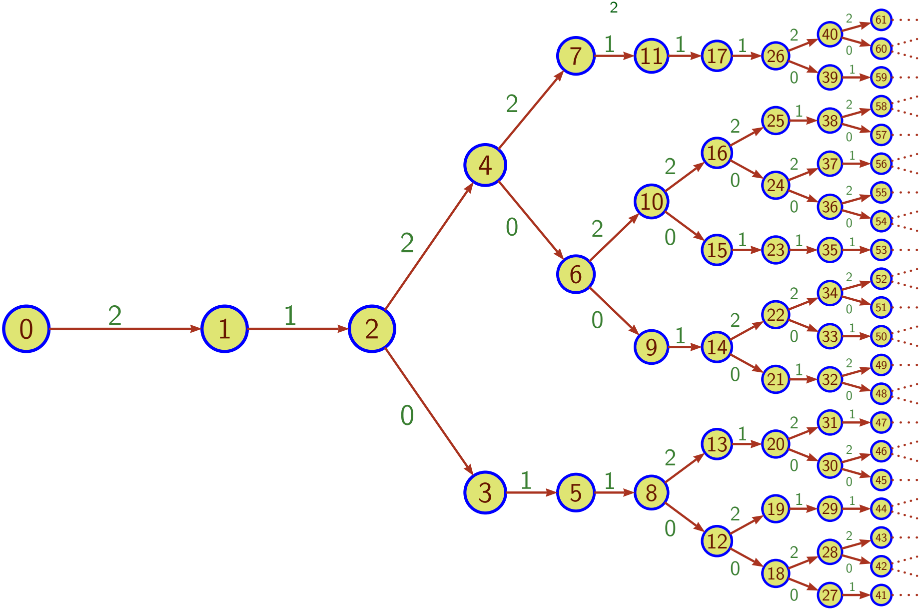
III

A property still missing a proper name

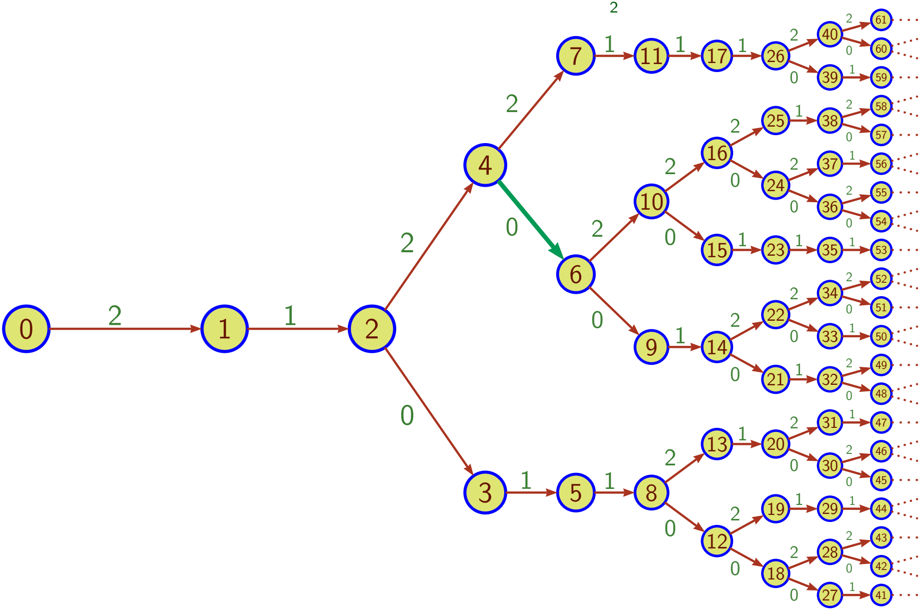
III

Another rabbit from the hat

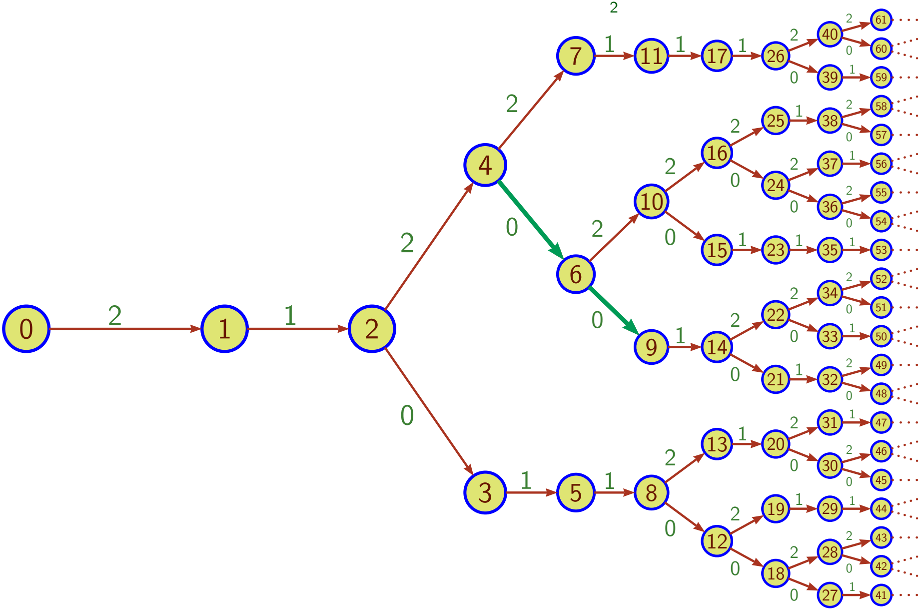
Minimal words in $T_{\frac{3}{2}}$



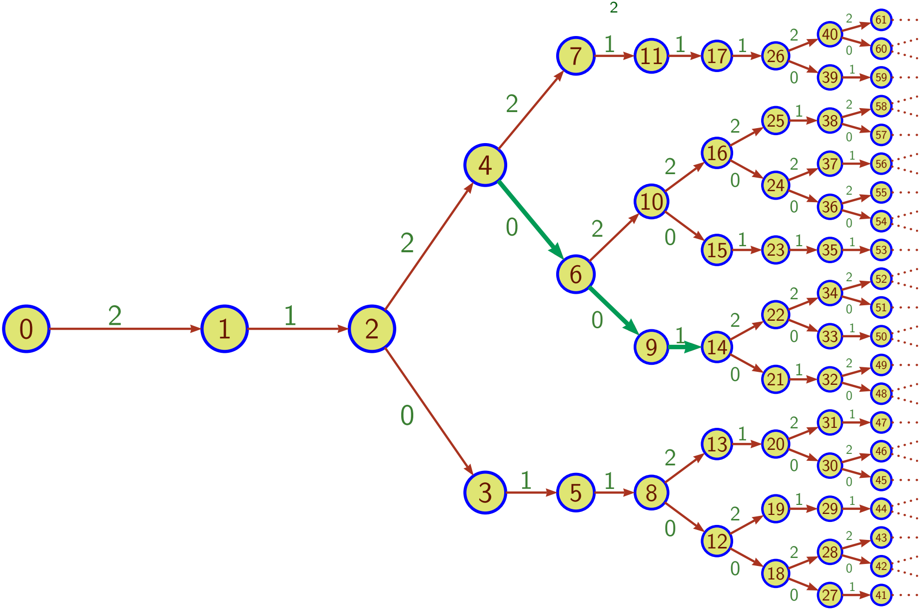
Minimal words in $T_{\frac{3}{2}}$



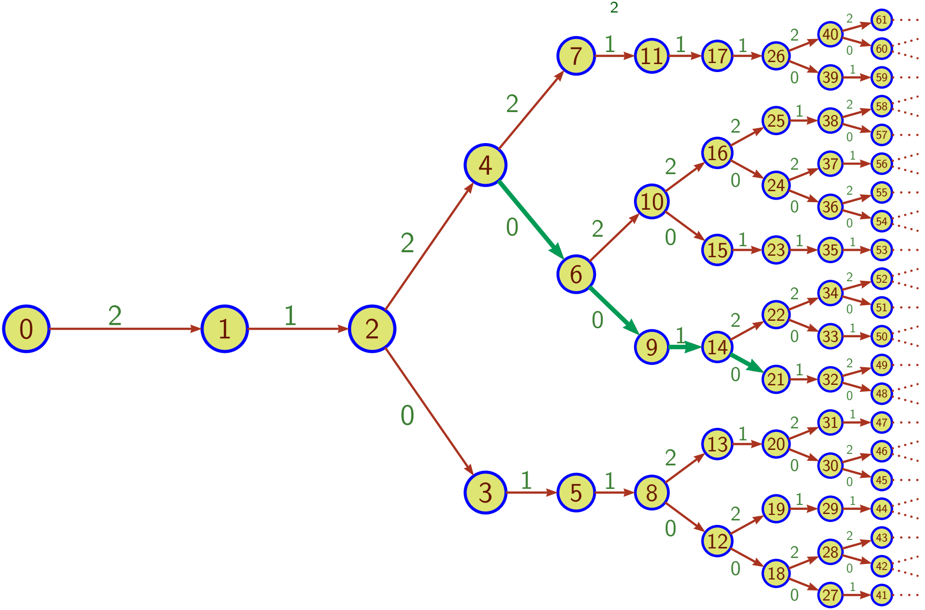
Minimal words in $T_{\frac{3}{2}}$



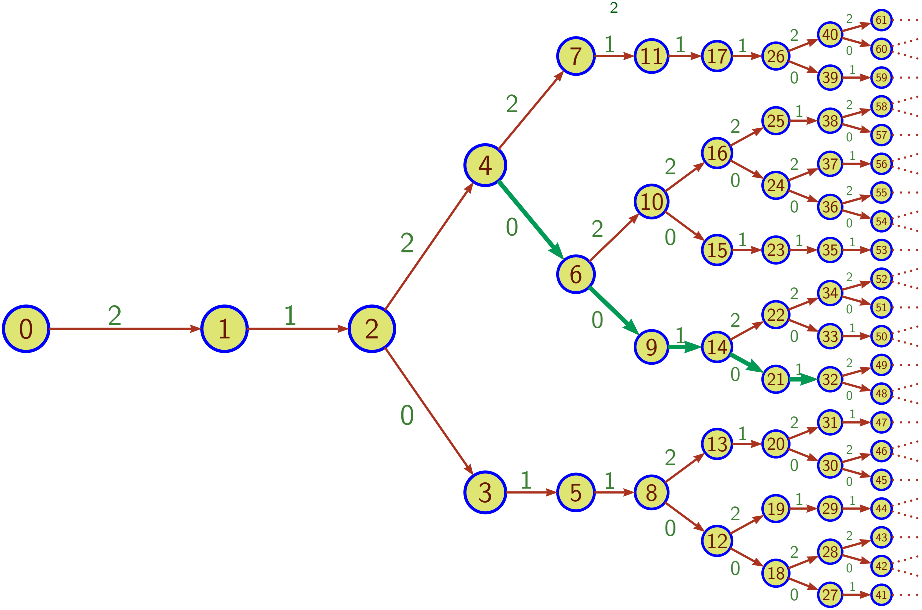
Minimal words in $T_{\frac{3}{2}}$



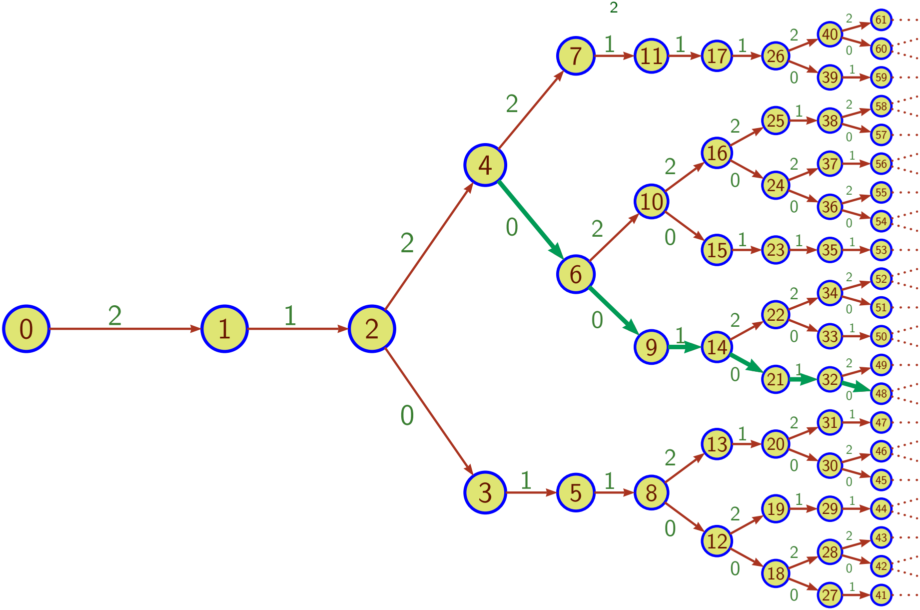
Minimal words in $T_{\frac{3}{2}}$



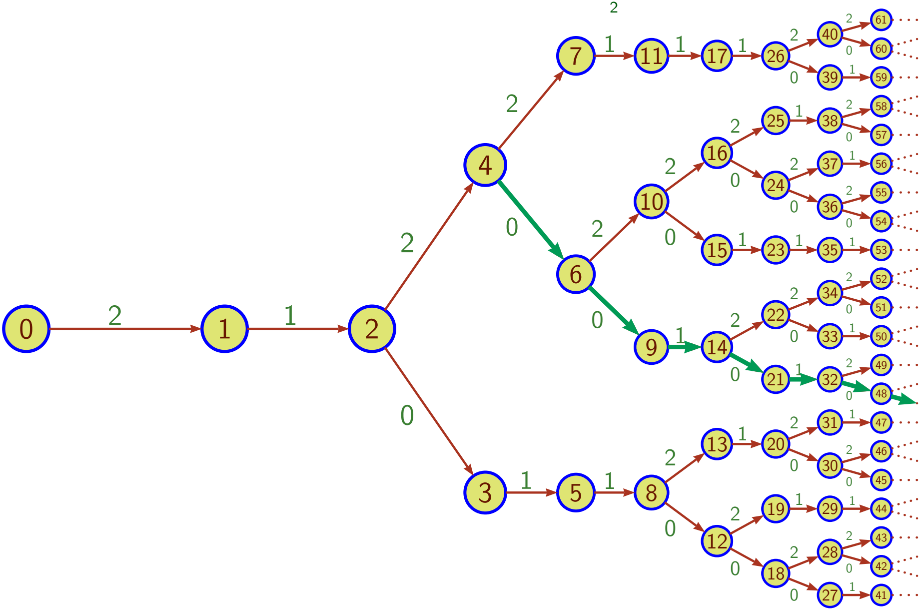
Minimal words in $T_{\frac{3}{2}}$



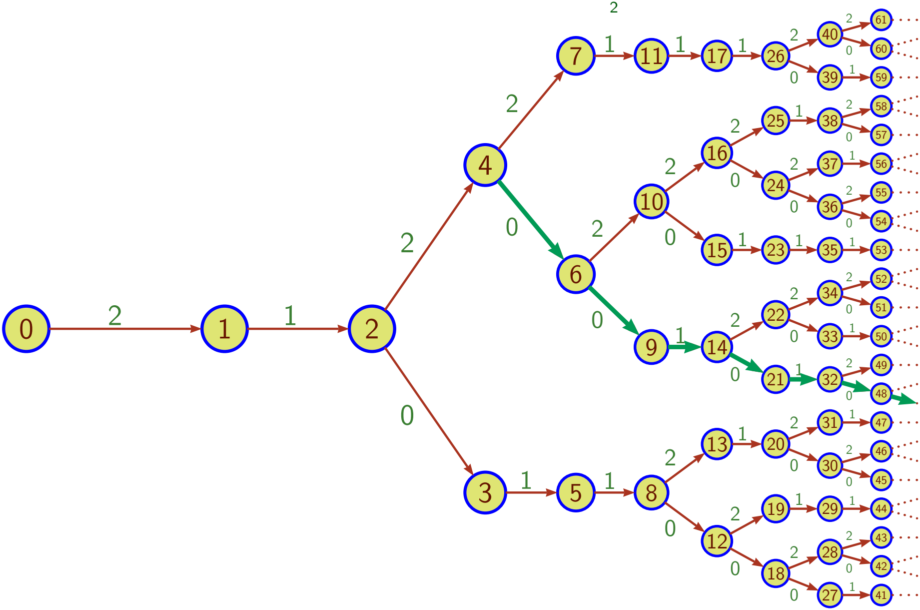
Minimal words in $T_{\frac{3}{2}}$



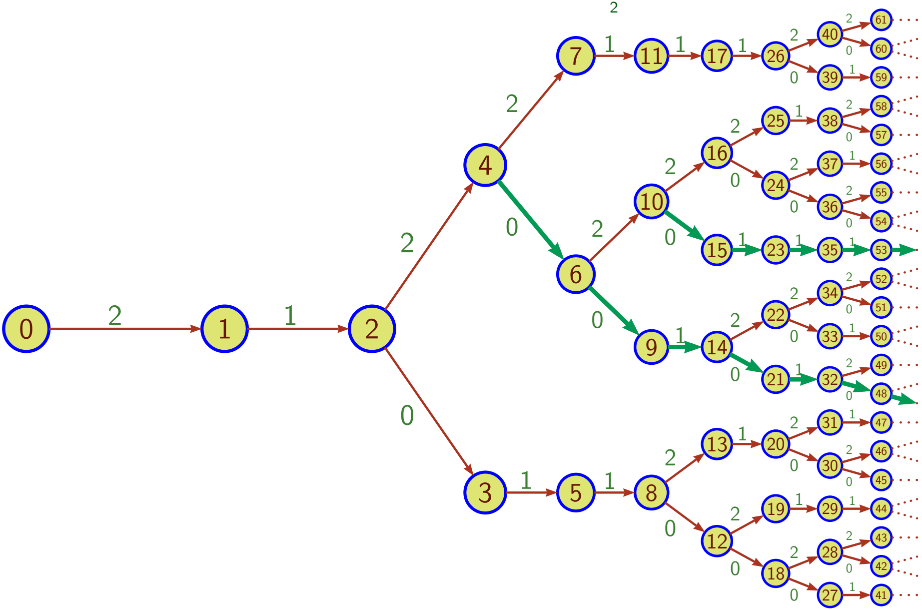
Minimal words in $T_{\frac{3}{2}}$



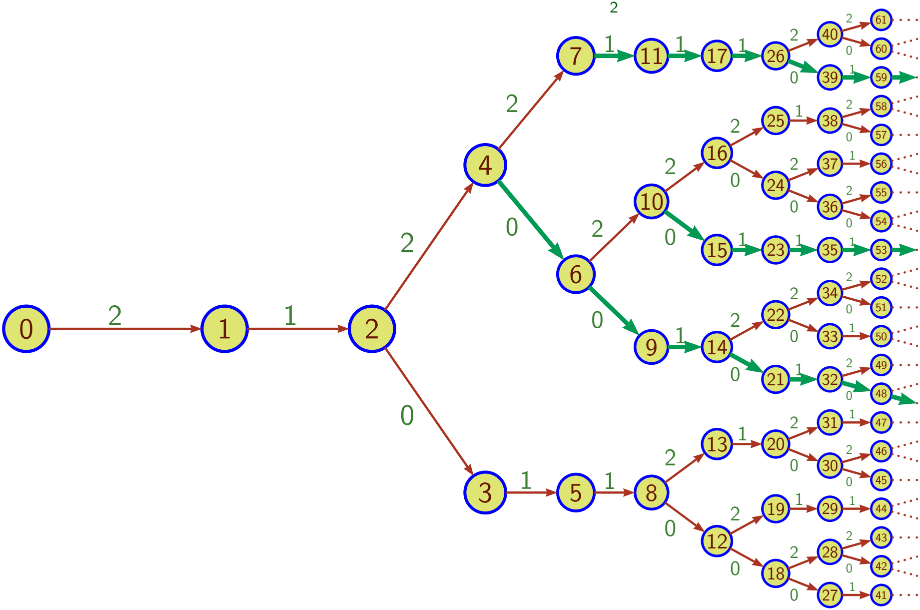
Minimal words in $T_{\frac{3}{2}}$



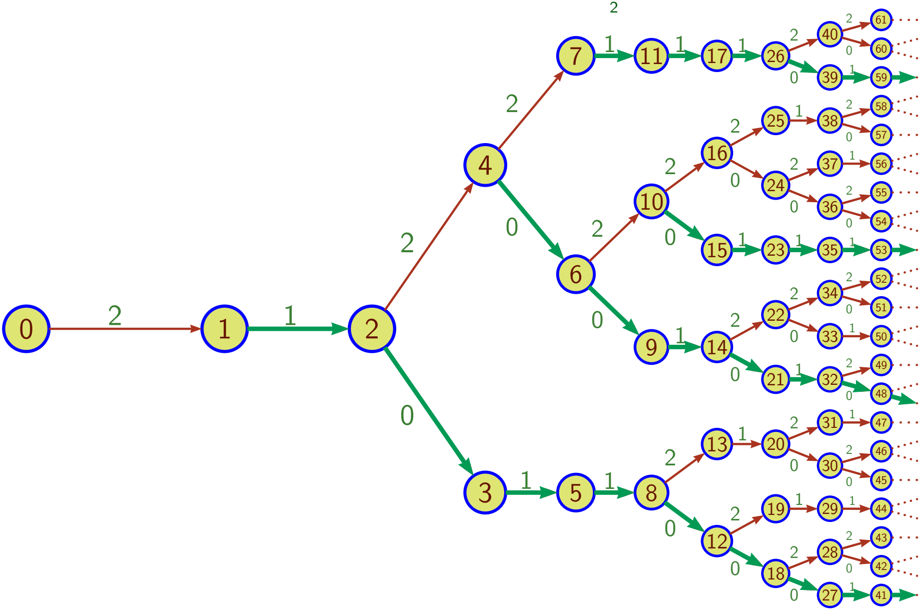
Minimal words in $T_{\frac{3}{2}}$



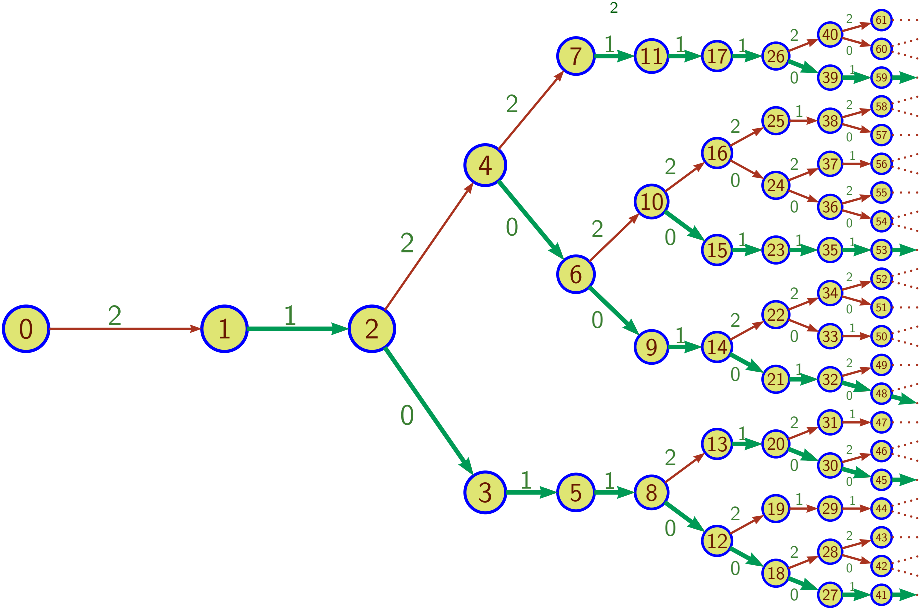
Minimal words in $T_{\frac{3}{2}}$



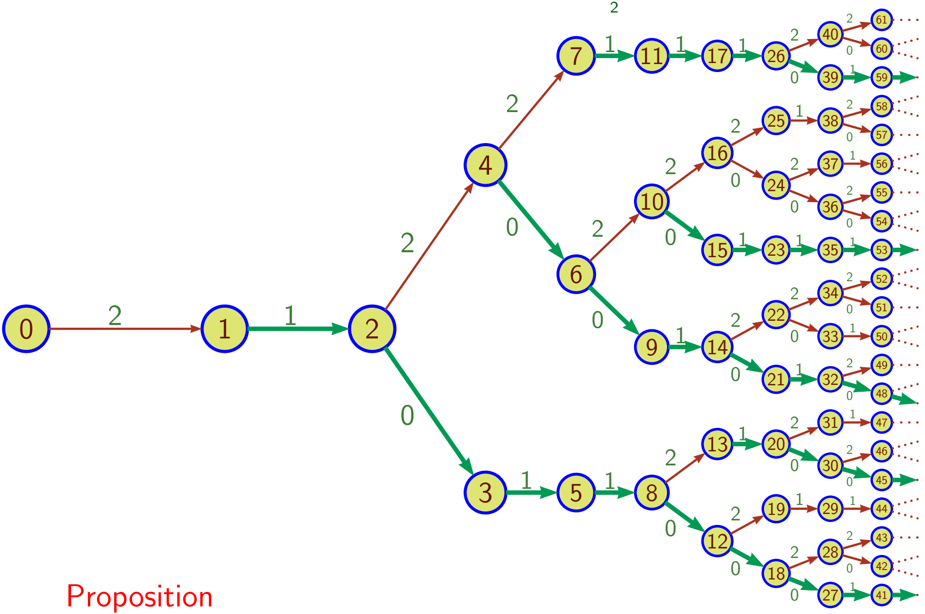
Minimal words in $T_{\frac{3}{2}}$



Minimal words in $T_{\frac{3}{2}}$



Minimal words in $T_{\frac{3}{2}}$



Proposition

The $\mathbf{w}_n^-$ *are all distinct words of* $\{0,1\}^\omega$.

Minimal words in $T_{\frac{3}{2}}$

Problem

What is the relation between the $\mathbf{w}_n^-$?

Minimal words in $T_{\frac{3}{2}}$

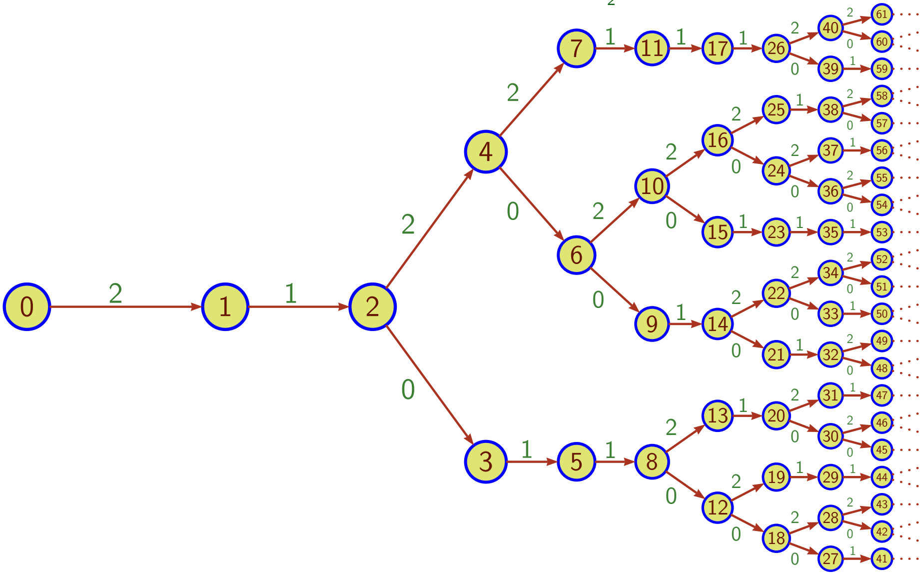
Problem

What is the relation between the $\mathbf{w}_n^-$?

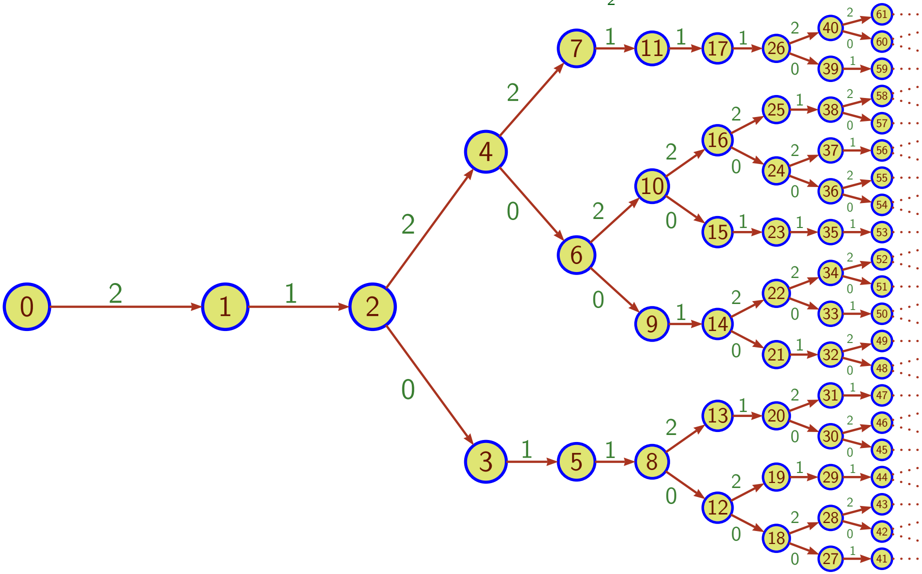
Conjecture

For every n there exists a finite transducer that takes $\mathbf{w}_n^-$ as input and outputs $\mathbf{w}_{n+1}^-$.

Derived transducer $\mathcal{D}_{\frac{3}{2}}$

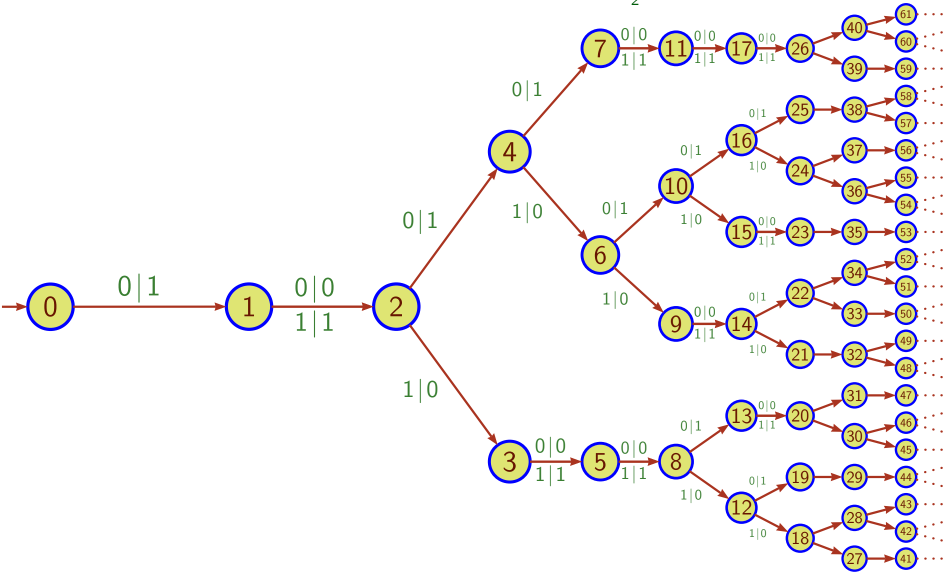


Derived transducer $\mathcal{D}_{\frac{3}{2}}$



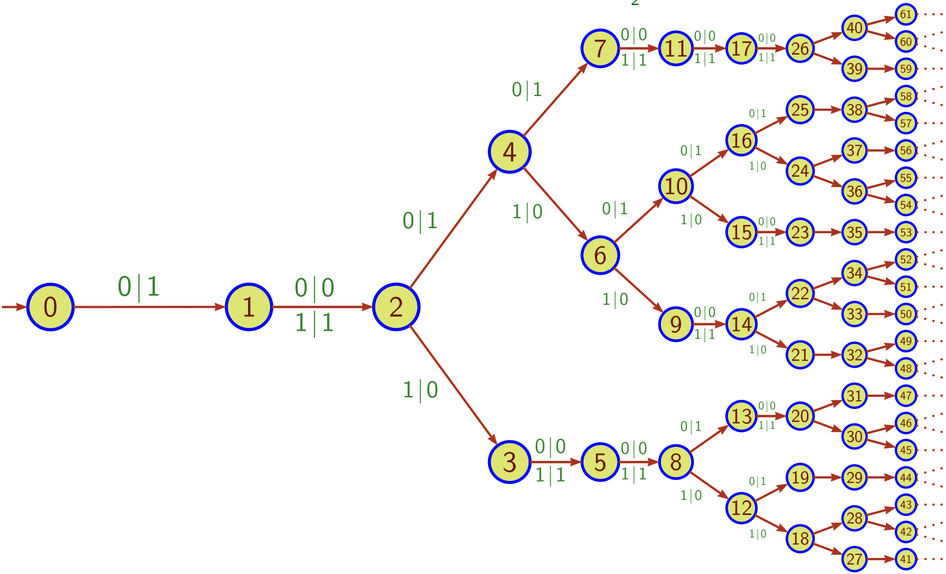
$2 \to 0\,|\,1 \qquad 1 \to 0\,|\,0, 1\,|\,1 \qquad 0 \to 1\,|\,0$

Derived transducer $\mathcal{D}_{\frac{3}{2}}$



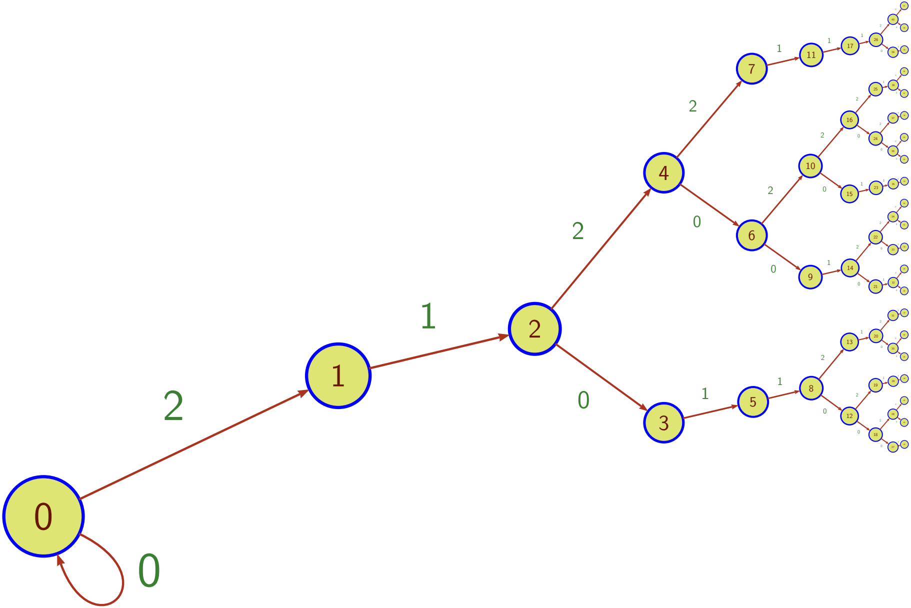
$2 \to 0|1 \qquad 1 \to 0|0, 1|1 \qquad 0 \to 1|0$

Derived transducer $\mathcal{D}_{\frac{3}{2}}$



Theorem

$$\forall n \in \mathbb{N} \qquad \mathcal{D}_{\frac{3}{2}}(\mathbf{w}_n^-) = \mathbf{w}_{n+1}^-$$



0
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9