Interval Translation Maps with Weakly Mixing Attractors

Silvia Radinger

based on a joint work with Henk Bruin

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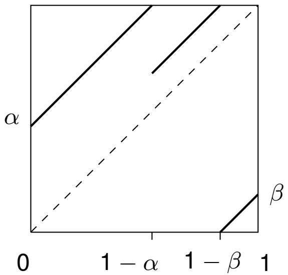
Two-parameter family of Interval Translation Maps (ITMs)

introduced by Bruin, Troubetzkoy in 2003

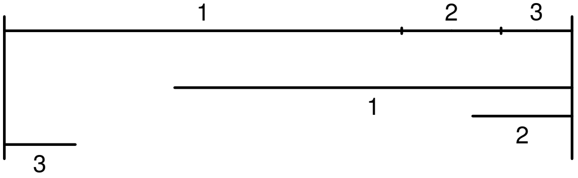
$$T_{\alpha,\beta}(x) = \begin{cases} x + \alpha, & x \in [0, 1-\alpha), \\ x + \beta, & x \in [1-\alpha, 1-\beta), \\ x - 1 + \beta, & x \in [1-\beta, 1] \end{cases}$$

on the parameter space $U = \{(\alpha, \beta) : 0 \leq \beta \leq \alpha \leq 1\}$.



Renormalization

Analyse the first return map to $[1 - \alpha, 1]$:



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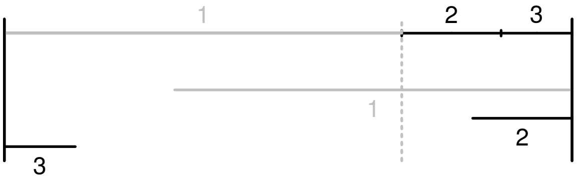

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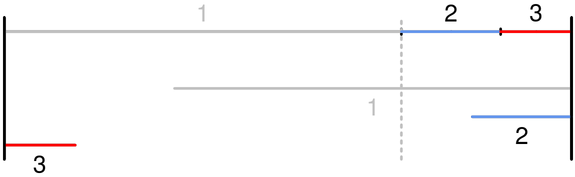

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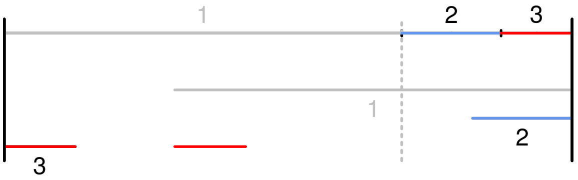

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$$(\alpha', \beta') = G(\alpha, \beta) = \left(\frac{\beta}{\alpha}\,,\ \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor\right).$$

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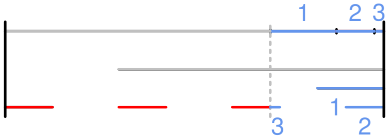

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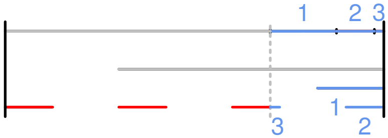
Two types of parameters

- **Finite Type:** $T_{\alpha,\beta}$ reduces to interval exchange transformation
- **Infinite Type:** $\Omega := \bigcap_{n \geq 0} \overline{T^n_{\alpha,\beta}([0,1])}$ is a Cantor set with $T_{\alpha,\beta}$ a minimal endomorphism.

The set of parameters (α, β) with $T_{\alpha,\beta}$ is of infinite type has Lebesgue measure zero.

S-adic Subshift



Symbolically, one renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}$$

with incidence matrix

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \det(A_k) = -1.$$

We define a S-adic subshift based on a sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N}$. The itinerary of the point 1 is

$$\rho = \lim_{i \to \infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

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Every *ITM of infinite type* in this family is uniquely characterised by a sequence $(k_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that

$k_{2i} > 1$ for infinitely many $i \in \mathbb{N}$ and $k_{2j-1} > 1$ for infinitely many $j \in \mathbb{N}$.

Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i\in\mathbb{N}}$ from an ITM of infinite type, is

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- aperiodic
- left-proper

Proof.

Left-proper.

$$\chi_{k_i} \circ \chi_{k_{i+1}} : \begin{cases} 1 \to 31^{k_i} \\ 2 \to 31^{k_i-1}2^{k_{i+1}} \\ 3 \to 31^{k_i-1}2^{k_{i+1}-1}. \end{cases}$$

□

Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

- aperiodic
- left-proper
- combinatorially recognizable

Proof.

Combinatorial Recognizability.
For example

$$\begin{aligned} x &= \ldots \mid 2 \mid 3\ 1\ 1 \mid 3\ 1 \mid 2 \mid 2 \mid 3\ 1\ 1 \mid \ldots \\ &= \ldots \chi_2(1)\ \chi_2(2)\ \chi_2(3)\ \chi_2(1)\ \chi_2(1)\ \chi_2(2) \ldots \end{aligned}$$

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Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

- aperiodic
- left-proper
- combinatorially recognizable
- primitive ($\Rightarrow (X, \sigma)$ is minimal).

Proof.

Primitivity.

$$\tilde{A}_i = \underbrace{A_1 \cdots A_1}_{r_{i,1} \geq 0} \cdot A_{k_{i,1}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,2} \text{ odd}} A_{k_{i,2}} \cdots \cdots A_{k_{i,m}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,m+1} \text{ even}} \cdot A_{k_{i,m+1}} \cdot A_{k_{i,m+2}},$$

is a full matrix for

- $k_{i,j} \geq 2$ for $1 \leq j \leq m+1$, $k_{i,m+2} \geq 1$.

□

Linearly Recurrent Subshift

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Theorem

The subshift (X, σ) associated to an ITM of infinite type is linearly recurrent if and only if

- $(k_i)_{i \in \mathbb{N}}$ *is bounded and*
- *the sets* $\{i : k_{2i} > 1\}$ *and* $\{i : k_{2i-1} > 1\}$ *have bounded gaps.*

Proof idea: Show $\exists$ telescoping $(\chi_{k_i})_i$ into finitely many, left-proper substitutions with full incidence matrices.

Weakly Mixing

Definition

A system (X, T) is called *weakly mixing* if the Koopman operator

$$U_T(f) = f \circ T$$

has 1 as its only eigenvalue.

If an eigenfunction f is

- in L^2, then its eigenvalue is called *measurable*,
- continuous, then its eigenvalue is called *continuous*.

In this talk we will restrict to continuous eigenvalues.

Eigenvalue Conditions - Periodic Case

Theorem (Host in 1986)

For a primitive substitution system a sufficient condition to have an eigenvalue $e^{2\pi it}$ for some $t \in (0,1)$ is

$$\sum_{n=1}^{\infty} \|\vec{t} A_{k_1} \cdots A_{k_n}\| < \infty, \qquad \vec{t} = (t, t, t),$$

where $\|x\|$ is the distance of a vector to the nearest integer lattice point.

This condition was later expanded to hold for linearly recurrent S-adic shifts and their continuous eigenvalues.

Lyapunov Exponents

Proposition

For every ITM of infinite type with sequence $(k_i)_{i\in\mathbb{N}}$, the infinite matrix multiplication $A_{k_1} \cdot A_{k_2} \cdots$ has two positive and one negative Lyapunov exponent.

There are independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$,

$$0 < \lambda_3 < 1 < |\lambda_2| < \lambda_1$$

and $C > 0$ such that for all n,

$$\|\vec{v}_i A_{k_1} \cdot A_{k_2} \cdots A_{k_n}\| \geq C|\lambda_i|^n \quad \text{for } i \in \{1,2\} \text{ and}$$
$$\|\vec{v}_3 A_{k_1} \cdot A_{k_2} \cdots A_{k_n}\| \leq C^{-1}\lambda_3^n.$$

Proof.

- A_k preserves positive octant $\mathcal{Q}^+$.
  By primitivity, $\tilde{A}_i$ is a positive integer matrix, thus $\log(\lambda_1) > 0$ and unstable space in interior of $\mathcal{Q}^+$.

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- A_k^{-1} preserves octant $\mathcal{Q}^- = \{x_1, x_2 \geq 0 \geq x_3\}$, change coordinates to $\mathcal{Q}^+$

$$B_k = UA_k^{-1}U^{-1} = \begin{pmatrix} 0 & 1 & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & k \end{pmatrix}$$

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- By

$$\log(\lambda_1) + \log(\lambda_2) + \log(\lambda_3) = 0$$

  if we can show $\log(\lambda_1) + \log(\lambda_3) < 0$, then $\log(\lambda_2) > 0$.

□

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Every ITM of infinite type with (pre-)periodic sequence $(k_i)_{i\in\mathbb{N}}$ *is weakly mixing.*

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Define $A^n = A_{k_1} \cdots A_{k_n} > 0$ with period n.

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- A^n is irreducible matrix, eigenvalues are cubic numbers.
- Stable space is one-dimensional in direction $E_3 = (u, v, -1)$.
- For $t(1,1,1)$ to be in integer translation of E_3:

$$\begin{pmatrix} t \\ t \\ t \end{pmatrix} + s \begin{pmatrix} u \\ v \\ -1 \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \text{ for } p, q, r \in \mathbb{Z} \text{ and reals } u, v > 0.$$

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- Find λ_3 is a quadratic number, a contradiction.

□

Eigenvalue Conditions - General Case

Theorem (Durand, Frank, Maass in 2019)

Let (X, σ) be a subshift based on a proper Bratteli diagram. Then $e^{2\pi it}$ is a continuous eigenvalue if and only if

$$\sum_{n=1}^{\infty} \max_{x \in X} |||\langle s_n(x), \vec{t}\tilde{A}_{k_1} \cdots \tilde{A}_{k_n} \rangle||| < \infty, \qquad \vec{t} = (t, t, t),$$

where

$$(s_n(x))_v = \sharp\{e \in E_{n+1} \ : \ e \succ x_{n+1}, s(e) = v\},$$

the vector $s_n(x)$ counts the number of incoming edges that are higher in the order than edge x_{n+1} in the path x.

Direction of the Stable Space

Follow $(u, v, 1-(u+v))B_k$ normalised to unit length, indicate the first two coordinates

$$H_k : (u, v) = \frac{1}{D_k}(v\ ,\ 1-v) \quad \text{for} \quad D_k = k(1-v) + 1 - u, \qquad (1)$$

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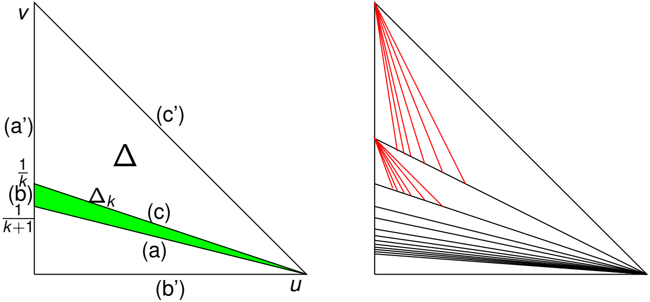
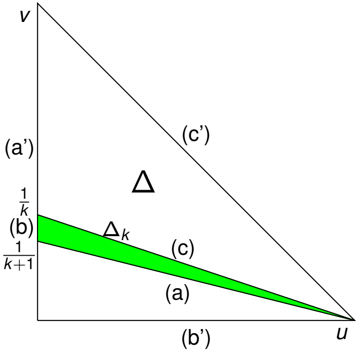


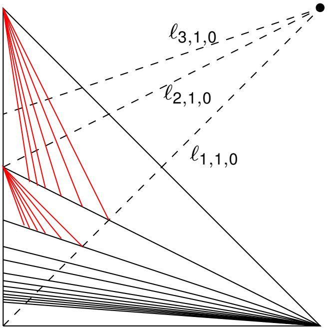
Figure: The simplex Δ and image $\Delta_k = H_k(\Delta)$ (left) and further images $H_{k_1}(H_{k_2}(\Delta))$ (right).

Direction of the Stable Space

To have $t(1,1,1)$ in stable direction

$$(u,v) \in \ell_{p,q,r} = \{(u,v) \in \Delta : u(q-r) = v(p-r) + q - p\}.$$





Results for Continuous Eigenvalues

Theorem

Let $T_{(\alpha,\beta)}$ be a ITM of infinite type with the stable space W^s. If $\vec{t} \notin W^s$, then $e^{2\pi it}$ is not a continuous eigenvalue of the Koopman operator.

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Theorem

There exist parameters (α, β) such that $\vec{t} \in W^s$ and $e^{2\pi it}$ is not a continuous eigenvalue of the Koopman operator.

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