On the binary digits of n and n^2

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joint work with Aloui, Jamet, Kaneko, Kopecki and Stoll

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- Generalized Ramanujan–Nagell equation: $y^n - D = x^2$, $D \neq 0$.
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  $\rightarrow$ Beukers (2002): At most four solutions for $D < 0$.
  $\rightarrow$ Bugeaud-Mignotte-Siksek (2006): All solutions for $1 \leq D \leq 100$.
- Catalan's conjecture (1844): $x^a - y^b = 1,\ a, b > 1,\ x, y > 0$
  $$\implies x = b = 3, y = a = 2.$$
  $\rightarrow$ Mihăilescu (2003) proved this conjecture.
- $\cdots$

Let $k \geq 2$

$$n^2 = 2^{a_{k-1}} + \cdots + 2^{a_1} + 1, \quad 0 < a_1 < \cdots < a_{k-1}. \tag{1}$$

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a, b positive integers.

- **Subadditive**: $s(a+b) \leq s(a) + s(b)$.
- **Submultiplicative**: $s(ab) \leq s(a)s(b)$.
- **2-additive**: If $b < 2^r$, $s(a2^r + b) = s(a) + s(b)$.

$$\begin{array}{ccccc} & (a)_2 & 0\cdots0 & 0\cdots0 & = a2^r \\ + & & 0\cdots0 & (b)_2 & = b \\ \hline & (a)_2 & 0\cdots0 & (b)_2 & = a2^r + b. \end{array}$$

The sum is **non-interfering**: no interaction between the digits of a and b.

Expected values:

$$\frac{1}{N}\sum_{1\leq n\leq N} s(n) = \frac{1}{2}\log_2(N) + O(1),$$

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$91 = 1+2+8+16+64$, $\quad (91)_2 = 1011011$, $\quad s(91) = 5$.
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Q: Are there finitely or infinitely many solutions for (2) ?

$$s(n) = s(n^2) = k, \qquad n \text{ odd.} \tag{2}$$

Theorem (Hare, Laishram, Stoll, 2011)

- If $1 \leq k \leq 8$, (2) has finitely many solutions.
- If $k = 12, 13$ or $k \geq 16$, (2) has infinitely many solutions.

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  Example: For $k = 5$, the set of solutions is $\{31, 79, 91, 157, 279\}$.

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  Example: For $k = 5$, the set of solutions is $\{31, 79, 91, 157, 279\}$.
- Give an infinite family of solutions for each k in the second case:

$$s(n) = s(n^2) = 12, \text{ for all } n = 111 \cdot 2^t + 111, \text{ with } t \geq 15.$$

$$\begin{array}{ccccccl} & & (111)_2 & 0\cdots0 & (111)_2 & = (n)_2 \\ (111^2)_2 & 0\cdots0 & (111^2)_2 & 00\cdots0 & (111^2)_2 & = (n^2)_2 \end{array}$$

And $s(111) = 6$, $s(111^2) = 4$.

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Theorem (Aloui, Jamet, Kaneko, Kopecki, P., Stoll, 2023)

If $9 \leq k \leq 11$, (2) has finitely many solutions.

Proof: new combinatorial tools and algorithms.

Summary

Write $n = 2^{\ell_m} x_m + \cdots + 2^{\ell_1} x_1 + x_0$ such that

$$(n)_2 = (x_m)_2 0 \cdots 0 (x_{m-1})_2 \cdots (x_1)_2 0 \cdots 0 (x_0)_2, \ \eta_i \geq 0.$$

$\rightarrow$ Not unique decomposition.

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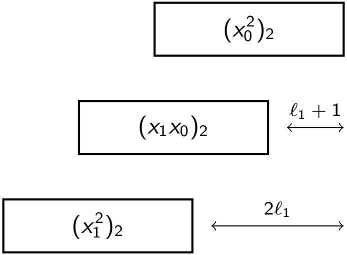
- For $m = 1$, $n^2 = 2^{2\ell_1} x_1^2 + 2^{\ell_1 + 1} x_1 x_0 + x_0^2$.



Interference graph

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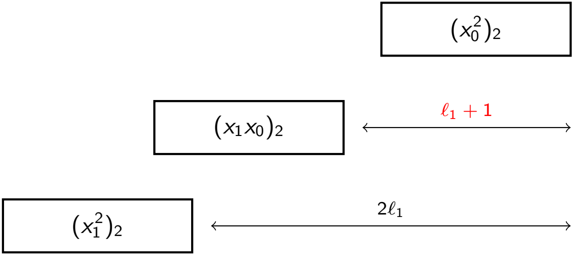

$|x|$ denotes the binary length of x.

If $\ell_1 + 1 > 2|x_0|$, no interference between $2^{\ell_1+1}x_1x_0$ and x_0^2.

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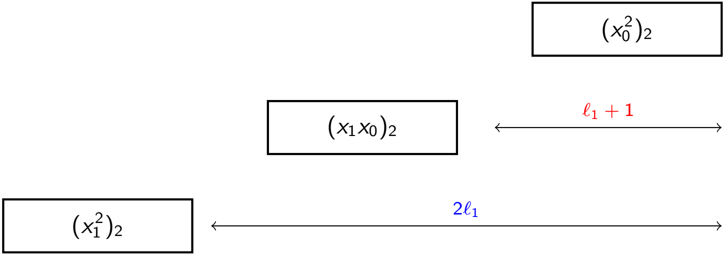

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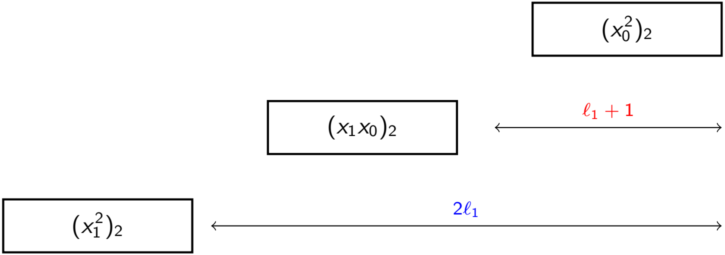

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In this case, n^2 is composed of three **independent** blocks.

- For $m = 2$, we have

$$n = 2^{\ell_2} x_2 + 2^{\ell_1} x_1 + x_0.$$
$$n^2 = 2^{2\ell_2} x_2^2 + 2^{\ell_2+\ell_1+1} x_2 x_1 + \underbrace{2^{2\ell_1} x_1^2 + 2^{\ell_2+1} x_2 x_0}_{\text{potential interference}} + 2^{\ell_1+1} x_1 x_0 + x_0^2.$$

Potential interference **even if** ℓ_i large enough.

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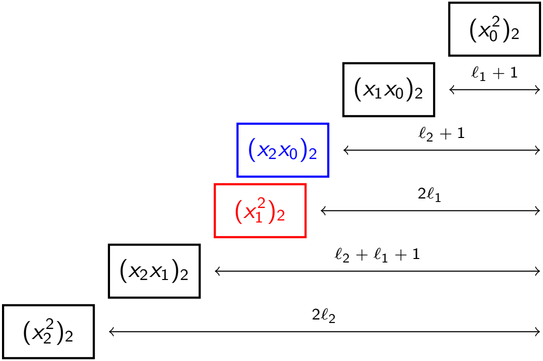

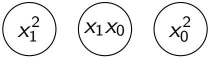

Figure: Interference graph for $m = 1$.

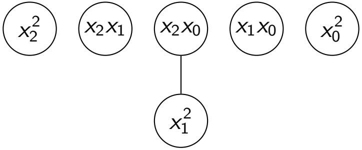

Figure: Interference graph for $m = 2$.

Interference graph

$m = 3$

- For $m = 3$, we have

$$n = 2^{\ell_3} x_3 + 2^{\ell_2} x_2 + 2^{\ell_1} x_1 + x_0.$$
$$n^2 = 2^{2\ell_3} x_3^2 + 2^{\ell_3+\ell_2+1} x_3 x_2 + \cdots + 2^{\ell_1+1} x_1 x_0 + x_0^2.$$

9 blocks and 5 potential interferences if ℓ_i large enough.

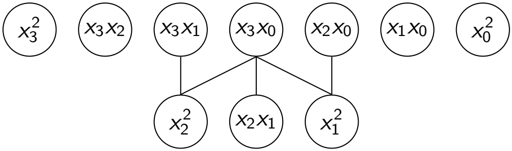

Figure: Interference graph for $m = 3$.

Factorization lemma

For $k \geq 1$, there exists N_k such that every odd integer $n \geq N_k$ with $s(n) = s(n^2) = k$ can be factorized

$$(n)_2 = (x_m)_2 0^{\eta_m} (x_{m-1})_2 \cdots (x_1)_2 0^{\eta_1} (x_0)_2,$$

with $\min(\eta_i) > 2\max(|x_i|) + k^2$.

Useful to

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- find easily infinite families of solutions.

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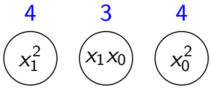
The bound N_k is very large: $N_9 = 2^{611\ 669}$.

Suppose n is such that $s(n) = s(n^2) = 11$ and satisfies the factorization lemma.
Distribute 11 1-bits in the 3 independent blocks. For example:



$$\begin{cases} s(x_1) + s(x_0) = 11, \\ s(x_1^2) = 4, \\ s(x_1x_0) = 3, \\ s(x_0^2) = 4. \end{cases}$$

Interference graph

Distribution for 11 bits: Example 1

Suppose n is such that $s(n) = s(n^2) = 11$ and satisfies the factorization lemma.

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4	3	4
x_1^2	x_1x_0	x_0^2

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Lemma (Kaneko, Stoll, 2022)

Let a, b be odd integers, $s(a) = \ell, s(b) = m \geq 3$.

$$s(ab) = 2 \implies ab < 2^{2\ell m - 4}.$$

$$s(ab) = 3 \implies ab < 2^{4\ell m - 13}.$$

Computer research is sufficient: $ab < 2^{107}$.

Interference graph

Distribution for 11 bits: Example 2

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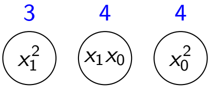

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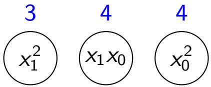

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Computer research is no longer possible for $s(x_1x_0) = 4$ since

Lemma (Kaneko, Stoll, 2022)

For all integers $L \geq 1$ there exist integers $\ell, m \geq L$ such that there are infinitely many pairs (a, b) of positive odd integers with

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Focus on solutions of $s(n^2) = k$ for small $k \geq 2$.

Summary

$$E_k := \left\{ n \in \mathbb{N} : s(n^2) = k, n \text{ odd} \right\}.$$

k	E_k
1	$\{1\}$
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	F infinite family.	for all $n \in F$, $s(n) = 2$.
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5	$F_1 \cup F_2 \cup F_3 \cup E_5'$,	Aloui, Jamet, Kaneko, Kopecki, P., Stoll (2023)
	F_i infinite families,	for all $n \in F_i$, $s(n) = 3$.
	E_5' finite set.	$\rightarrow$ Combinatorial tools.

$E_k := \left\{ n \in \mathbb{N} : s(n^2) = k, n \text{ odd} \right\}$.

k	E_k	
1	$\{1\}$	
2	$\{3\}$	
3	$\{7, 23\} \cup F$,	Szalay (2002).
	F infinite family.	for all $n \in F$, $s(n) = 2$.
		$\rightarrow$ Beukers result on the RN equation.
4	Finite set.	Bennett, Bugeaud, Mignotte (2012).
		$\rightarrow$ Linear forms in logarithms.
	$\{13, 15, 47, 111\}$	Conjecture (2012), **still open**.
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	F_i infinite families,	for all $n \in F_i$, $s(n) = 3$.
	E_5' finite set.	$\rightarrow$ Combinatorial tools.
	$E_5' = \{29, \ldots, 5793\}$.	Conjecture (2023)

3	4	4
x_1^2	x_1x_0	x_0^2

$$\begin{cases} s(x_1) + s(x_0) = 11, \\ s(x_1^2) = 3, \\ s(x_1x_0) = 4, \\ s(x_0^2) = 4. \end{cases}$$

Problem: All solutions of $s(x_0^2) = 4$ are not known.

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But, we only need integers of E_4 with **bounded** sum of digits.

$$E_{k,\lambda} := \left\{ n \in \mathbb{N} : s(n^2) = k, s(n) = \lambda,\ n \text{ odd} \right\}, \quad E_k = \bigcup_{\lambda \geq 1} E_{k,\lambda}.$$

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Theorem (Aloui, Jamet, Kaneko, Kopecki, P., Stoll, 2023)

$$\bigcup_{1\leq\lambda\leq 17} E_{4,\lambda} = \{13, 15, 47, 111\}.$$

Proof: By an algorithm that constructs all possible solutions for a given weight.

Supports the conjecture $E_4 = \{13, 15, 47, 111\}$ since $s(111)=6$.

Few binary digits

Distribution for 11 bits: Example 2

$$\overset{3}{x_1^2} \qquad \overset{4}{x_1x_0} \qquad \overset{4}{x_0^2}$$

$$\begin{cases} s(x_1)+s(x_0)=11, \\ s(x_1^2)=3, \\ s(x_1x_0)=4, \\ s(x_0^2)=4. \end{cases}$$

$$\implies \begin{cases} s(x_0)+s(x_1)=11 \\ x_1\in\{7,23\}, \text{ or } x_1=2^\ell+1, \ell\geq 2. \\ s(x_1x_0)=4, \\ x_0\in\{13,15,47,111\}. \end{cases}$$

Then $s(x_0)+s(x_1)\leq 4+6<11 \implies$ no solution for **this** distribution of digits.

2	4	5
x_1^2	x_1x_0	x_0^2

$$\begin{cases} s(x_1)+s(x_0)=11, \\ s(x_1^2)=2, \\ s(x_1x_0)=4, \\ s(x_0^2)=5. \end{cases}$$

Same problem for solutions of $s(x_0^2)=5$.

Theorem (Aloui, Jamet, Kaneko, Kopecki, P., Stoll, 2023)

$$\bigcup_{4\leq\lambda\leq 15} E_{5,\lambda} = \{29, 31, 51, 79, 91, 95, 157, 223, 279, 479, 727, 1471, 5793\}.$$

$\rightarrow$ This set is the conjectured set for E'_5.

$$\begin{array}{ccc} 2 & 4 & 5 \\ x_1^2 & x_1x_0 & x_0^2 \end{array} \qquad \begin{cases} s(x_1)+s(x_0)=11, \\ s(x_1^2)=2, \\ s(x_1x_0)=4, \\ s(x_0^2)=5. \end{cases}$$

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$\rightarrow$ This set is the conjectured set for E_5'.

$$\Longrightarrow \begin{cases} s(x_0)+s(x_1)=11, \\ x_1=3, \\ s(x_1x_0)=4, \\ x_0\in\{29,31,\dots,1471,5793\}. \end{cases} \Longrightarrow s(x_0)=9 \text{ and } x_0=1471.$$

Since $s(3\cdot 1471)=7>4$, there is **no solution** for this distribution of digits.

Theorem (Aloui, Jamet, Kaneko, Kopecki, P., Stoll, 2023)

If $9 \leq k \leq 11$, (2) has finitely many solutions.

Proof

- Fix k and consider n that satisfies the factorization lemma for some $m \leq k$.
- Finite number of distribution of digits for each m.
- Prove that all of them leads to a contradiction.

Summary

Suppose that $n = 1 + 2^\ell y$, y odd, $\ell \geq 1$, such that $s(n^2) = 4$.

$$s(1 + 2^{\ell+1} y + 2^{2\ell} y^2) = 4,$$
$$s(y + 2^{\ell-1} y^2) = 3.$$

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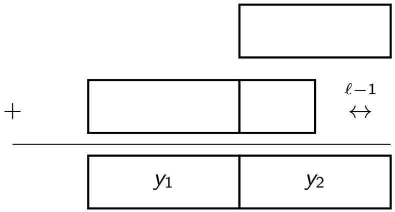

$= (y)_2$

$= (y^2)_2$

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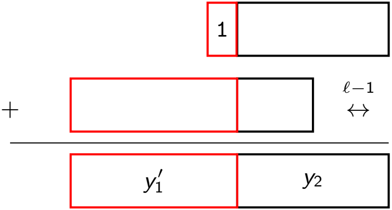

$= (1y)_2$

$= ((1y)^2)_2$

$= (1y + 2^{\ell-1}(1y)^2)_2$

Extending y with 1 does not changed y_2.

For odd integers a, b,

$$a \equiv b \pmod{2^{\lambda}} \implies a^2 \equiv b^2 \pmod{2^{\lambda+1}}.$$

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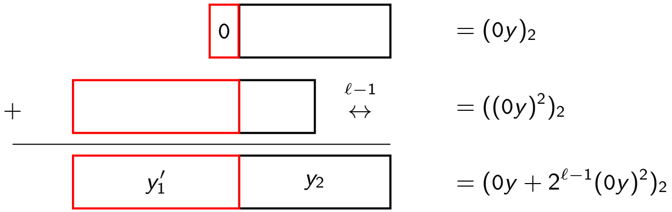

Extending y with 0 does not changed y_2.

For odd integers a, b,

$$a \equiv b \pmod{2^\lambda} \implies a^2 \equiv b^2 \pmod{2^{\lambda+1}}.$$

Start from a candidate y and extend y on the left by

- a 1: finite number of extension since $s(y) \leq k - 1$ by **hypothesis**.
- a 0: not a **too large** block of consecutive 0, otherwise too many digits in the sum.

$\Longrightarrow$ Finite number of possible extensions.

If the algorithm ends, it gives all solutions to $s(n^2) = 4$ and $s(n) = k$.

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15	1 sec
16	102 sec
17	2h50 mn

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Summary

$$s(n) = s(n^2) = k, \qquad n \text{ odd.} \tag{2}$$

k	1–8	9–11	12–13	14–15	$\geq$16
Solutions	$< \infty$	$< \infty$	∞	?	∞

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For $k = 14, 15$, (2) has infinitely many solutions.

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For $k = 14, 15$, (2) has infinitely many solutions.

$\rightarrow$ Global research of every odd integer n such that $s(n) = s(n^2) = k, \quad n \leq 2^{80}$.

Number of integers to check: $\binom{79}{k-1}$ **very large**.

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Parallelize the program.

Set up the first four nonzero bits of n:

$$n = 1 + 2^a + 2^b + 2^c + y, \;\; 1 \leq a < b < c, \;\; 2^c < y \leq 2^{80}.$$

Number of integers to check: $\binom{79-c}{k-4}$ **smaller** but **large** number of cases.

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- For $k = 11$, the largest solution is $n = 35463511416833$ of binary length 46.
- For $k = 14, 15$, we have solutions of binary length 80, for example:

  $n = 605643510452789079965697$ satisfies $s(n) = s(n^2) = 14$.

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Conjecture

For $k = 14, 15$, (2) has finitely many solutions.

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Thank you for your attention !