Arithmetics in alternate base numeration systems

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Christiane Frougny, Český Krumlov 2007

Émilie Charlier, Linz 2007

Cantor real numeration systems

$\mathcal{B} = (\beta_k)_{k \geq 1}$ with $\beta_k > 1$, such that $\lim\limits_{k \to \infty} \prod_{i=1}^{k} \beta_i = +\infty$

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We have $d^*(\mathcal{B},1) = \lim\limits_{x\to 1-} d(\mathcal{B},x)$.

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Theorem (Parry 1960, Charlier & Cisternino 2021)

An integer sequence $\mathbf{x} = (x_k)_{k \geq 1} \in D_{\mathcal{B}}$ *if and only if for all* $i \in \mathbb{N}$

$$0^\omega \preceq_{lex} \sigma^i(\mathbf{x}) \prec_{lex} d^*(\sigma^i(\mathcal{B}), 1).$$

shift map σ: $\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 x_4 \cdots$

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Theorem (Bertrand-Mathis 1986)

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Perron: dominant eigenvalue of a non-negative irreducible integer matrix
Pisot: algebraic integer > 1 with conjugates in the unit circle

Alternate bases $\mathcal{B} = (\beta_1, \ldots, \beta_p)$

$\mathcal{B}$-shift $\Sigma_{\mathcal{B}} = \overline{\bigcup_{i=0}^{p-1} D_{\sigma^i(\mathcal{B})}}$

Theorem (Charlier & Cisternino 2021)

$\mathcal{B}$-shift is sofic iff $\mathcal{B}$ is a **Parry alternate base***,*
i.e. $d(\sigma^i(\mathcal{B}), 1)$ is eventually periodic for all $i \in \{0, 1, \ldots, p-1\}$.

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Theorem (Charlier,Cisternino,M.,P. 2022)

If $\mathcal{B}$ is a Parry alternate base, then $\delta = \prod_{i=1}^{p} \beta_i$ is an algebraic integer and $\beta_i \in \mathbb{Q}(\delta)$.

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If δ is Pisot and $\beta_i \in \mathbb{Q}(\delta)$, then $\mathcal{B}$ is a Parry alternate base.

Arithmetics on β-expansions: finiteness property

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Finite beta-expansions

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Abstract. We characterize numbers having finite β-expansions where β belongs to a certain class of Pisot numbers: when the β-expansion of 1 is equal to $a_1a_2 \dots a_m$, where $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and when the β-expansion of 1 is equal to $t_1t_2 \dots t_m(t_{m+1})^\omega$ where $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} \geq 1$.

Finiteness and positive finiteness

Finite representation: finitely many non-zeros

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$\mathcal{B} = (\beta)$: $\quad \mathrm{Fin}(\mathcal{B}) = \bigcup_{k\in\mathbb{N}} \left\{ \beta^k x : |x| \in [0,1),\ d(\mathcal{B}, |x|) \text{ is finite} \right\}$

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Property F:
$\mathrm{Fin}(\mathcal{B})$ is closed under addition.

Property PF:
$\mathrm{Fin}(\mathcal{B})$ is closed under addition of positive elements.

Necessary conditions for F if $p = 1$

Theorem (Frougny & Solomyak 1992)

Let $\beta > 1$ satisfy PF. Then

- β *is a Pisot number;*
- β *has no conjugate in* $(0, 1)$.

If, moreover, β *has F, then* $d(\beta, 1)$ *is finite.*

Classes of bases with PF and F for $p = 1$

Theorem (Frougny & Solomyak 1992)

Let $\beta > 1$ and $d(\beta, 1) = t_1 t_2 t_3 \cdots$.

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Other classes of bases with F (Hollander, Akiyama, ...)

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- *If moreover* $d(\beta, 1)$ *is finite, then* β *satisfies F.*

Other classes of bases with F (Hollander, Akiyama, ...)

Theorem (Akiyama 2006)

Let $\beta > 1$ *not satisfy F, denote* $d(\beta, 1) = t_1 t_2 t_3 \cdots$.

- *If* β *satisfies PF, then* $t_1 \geq t_2 \geq t_3 \geq \cdots$.

Necessary conditions for F if $p > 1$

Alternate base $\mathcal{B} = (\beta_1, \ldots, \beta_p)$, $\delta = \prod_{i=1}^{p} \beta_i$

Let $\mathcal{B}$ satisfy F or PF, then $\sigma^i(\mathcal{B})$ satisfies F or PF, respectively.

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Theorem (M., P., Studeničová 2023)

Let $\mathcal{B}$ satisfy PF. Then

- *δ is a Pisot or a Salem number;*
- *$\beta_i \in \mathbb{Q}(\delta)$ for all i;*
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- *If, moreover, $\mathcal{B}$ has F, then $d(\sigma^i(\mathcal{B}), 1)$ is finite for all i.*

Classes of bases with PF and F for $p > 1$

Theorem (M., P., Studeničová)

Let $d(\sigma^\ell(\mathcal{B}), 1) = t_1^{(\ell)} t_2^{(\ell)} t_3^{(\ell)} \cdots$ *satisfy*

$$t_1^{(\ell)} \geq \; t_2^{(\ell-1)} \geq \; t_3^{(\ell-2)} \geq \; \cdots \qquad \text{for every} \;\; \ell \in \{0, 1, \ldots, p-1\}.$$

Then $\mathcal{B}$ *is a Parry alternate base and has PF.*
Moreover, if $d(\sigma^\ell(\mathcal{B}), 1)$ *are finite then* $\mathcal{B}$ *has F.*

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Then $\mathcal{B}$ *is a Parry alternate base and has PF.*
Moreover, if $d(\sigma^\ell(\mathcal{B}), 1)$ *are finite then* $\mathcal{B}$ *has F.*

$$d(\mathcal{B}, 1) = a_1 a_2 \cdots, \qquad d(\sigma(\mathcal{B}), 1) = b_1 b_2 \cdots, \qquad d(\sigma^2(\mathcal{B}), 1) = c_1 c_2 \cdots$$

$$a_1 \geq c_2 \geq b_3 \geq a_4 \geq c_5 \geq b_6 \geq \cdots$$

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Arithmetics: Periodicity

Alternate base $\mathcal{B} = (\beta_1, \ldots, \beta_p)$, $\delta = \prod_{i=1}^{p} \beta_i$

$$\mathrm{Per}(\mathcal{B}) = \bigcup_{k \in \mathbb{N}} \left\{ \delta^k x : |x| \in [0,1),\ d(\mathcal{B}, |x|) \text{ is eventually periodic } \right\}$$

$\mathrm{Per}(\mathcal{B}) \subset \mathbb{Q}(\beta_1, \ldots, \beta_p)$. How about opposite inclusion?

Periodicity for $p = 1$

Theorem (Schmidt 1980)

Let $\beta > 1$.

1. *If* $\mathbb{Q} \subset \mathrm{Per}(\beta)$, *then* β *is either a Pisot or a Salem number.*
2. *If* β *is a Pisot number, then* $\mathrm{Per}(\beta) = \mathbb{Q}(\beta)$.

Periodicity for $p > 1$

Theorem (Charlier, Cisternino & Kreczman 2022)

Let $\mathcal{B} = (\beta_1, \beta_2, \ldots, \beta_p)$ be an alternate base, $\delta = \prod_{j=1}^{p} \beta_j$.

1. *If $\mathbb{Q} \subset \mathrm{Per}(\sigma^j(\mathcal{B}))$ for all $j \in \mathbb{N}$, then δ is either a Pisot or a Salem number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$.*
2. *If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$, then $\mathbb{Q}(\delta) = \mathrm{Per}(\sigma^j(\mathcal{B}))$ for all $j \in \mathbb{N}$.*

Rationals with purely periodic expansion

Schmidt: For β root of $x^2 - ax - 1$, $a \geq 1$, every rational in $[0, 1)$ has purely periodic β-expansion.

$d(\tau, \frac{1}{2}) = (010)^\omega$

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Hama & Imahashi: For β root of $x^2 - ax + 1$, $a \geq 3$, no rational in $[0, 1)$ has purely periodic β-expansion.

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Question: For which $\mathcal{B}$

$$\gamma(\mathcal{B}) = \sup \left\{ \nu : \forall\, x \in [0,\nu) \cap \mathbb{Q},\ d(\mathcal{B},x) \text{ purely periodic } \right\} > 0?$$

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Necessary condition: $\gamma(\mathcal{B}) > 0 \implies \beta$ Pisot unit

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If β is a Pisot unit with F, then $\gamma(\mathcal{B}) > 0$.

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Conjecture

If $\gamma(\mathcal{B}) > 0$, then β is a Pisot unit with F.

Holds for

- β quadratic (Schmidt 1980, Hama & Imahashi 1997)

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Holds for

- β quadratic (Schmidt 1980, Hama & Imahashi 1997)
- β cubic (Adamczewski, Frougny, Siegel & Steiner 2010)

Rationals with purely periodic expansion for $p > 1$

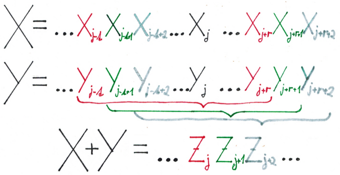
Let $\mathcal{B} = (\beta_1, \ldots, \beta_p)$ and $\delta = \prod_{i=1}^{p} \beta_i$.

Theorem (M. & P.)

If $\mathcal{B}$ satisfies F and δ is a Pisot unit, then $\gamma(\mathcal{B}) > 0$.

Redundant alphabets

Avizienis 1961: parallel addition



t, r - fixed and $z_j = \Phi(x_{j-t}+y_{j-t},\ \ldots\ , x_j+y_j,\ \ldots\ , x_{j+r}+y_{j+r})$

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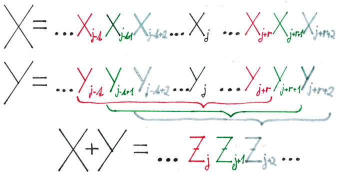

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$$\mathcal{A} = \{-1, 0, \ldots, b-1\} \quad \text{or} \quad \mathcal{A} = \{0, \ldots, b-1, b\}$$

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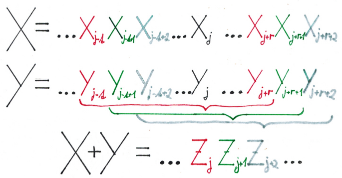

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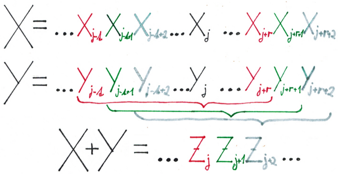

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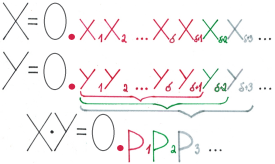
Frougny, P. & Svobodová 2011, 2013, 2014: characterization of $\beta \in \mathbb{C}$ allowing parallel addition and the minimal size of $\mathcal{A}$

Redundant alphabets

Trivedi & Ercegovac 1977: on-line multiplication and division



fixed delay $\delta \in \mathbb{N}$ and $p_n = \Phi(x_1, y_1, \ \ldots, \ x_{n+\delta}, y_{n+\delta})$

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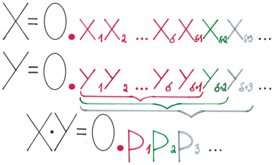

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Frougny & Surarerks 2003: on-line multiplication for the bases: real $\beta > 1$, $-b$ and $i\sqrt{b}$ with $b \in \mathbb{N}$

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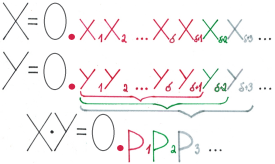

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Pavelka, Frougny, P. & Svobodová 2019: multiplication and division for $\beta \in \mathbb{C}$ allowing parallel addition

From webpage of Christiane

17 papers of Christiane have Prague coauthors! (since 1998)

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Thank you, Christiane!

Christiane Frougny, Český Krumlov 2007