Using automata to realise additions

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CNRS and Université Gustave-Eiffel

Special session in honor of Christiane Frougny's 75th birthday
Numeration 2023, Liège, Belgium

2023–05–23

1 Warm-up: Addition in integer base

Integer base

Definition

$p > 1$: an integer

$$\mathrm{Val}(a_n \cdots a_1 a_0) = a_n p^n + \cdots + a_1 p + a_0 = \sum_{i=0}^{n} a_i p^i$$

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Example: $p = 2$

$$\mathrm{Val}(0) = \mathrm{Val}(\varepsilon) = 0 \qquad (1)$$

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Nonstandard digits

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Nonstandard digits

Negative digits

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{ccccccc}
 & 0 & 1 & 1 & 1 & 0 & 0 \\
+ & 0 & 1 & 0 & 1 & 1 & 0 \\
= & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Addition is realised by a (multi-tape) automaton

$$
\begin{array}{cccccccl}
 & & & & & & 0 & \leftarrow \text{carry} \\
 & 0 & 1 & 1 & 1 & 0 & 0 & \\
+ & 0 & 1 & 0 & 1 & 1 & 0 & \\
= & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}
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= & \cdot & \cdot & \cdot & \cdot & 1 & 0
\end{array}
$$

Addition is realised by a (multi-tape) automaton

			1	0	0	0	← carry
	0	1	1	1	0	0	
+	0	1	0	1	1	0	
=	·	·	·	0	1	0	

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$=$	$\cdot$	1	0	0	1	0	

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		0	1	1	1	0	0
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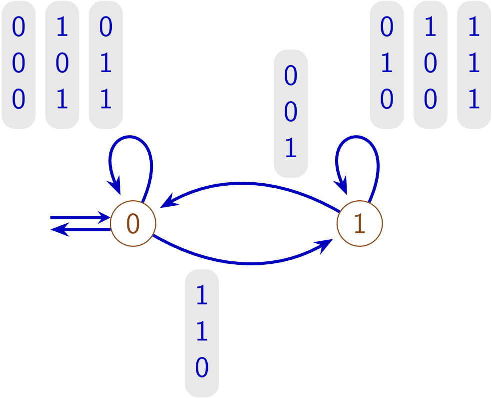
$=$		1	1	0	0	1	0

Addition is realised by a (multi-tape) automaton

	0	1	1	1	0	0	0 ← state
		0	1	1	1	0	0
+		0	1	0	1	1	0
=		1	1	0	0	1	0



Addition is realised by a (multi-tape) automaton

0 ← state

	0	1	1	1	0	0
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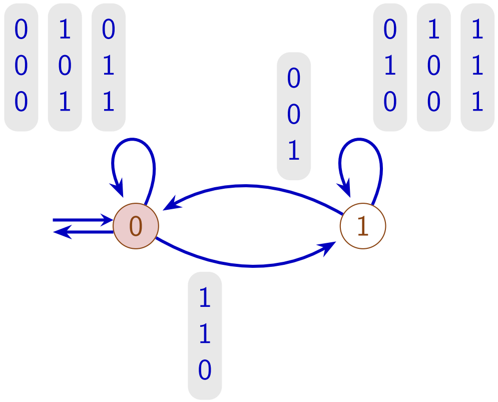

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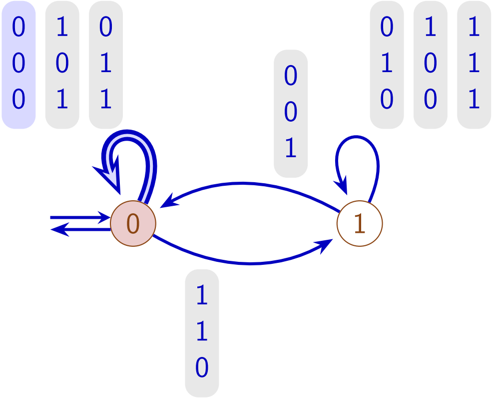

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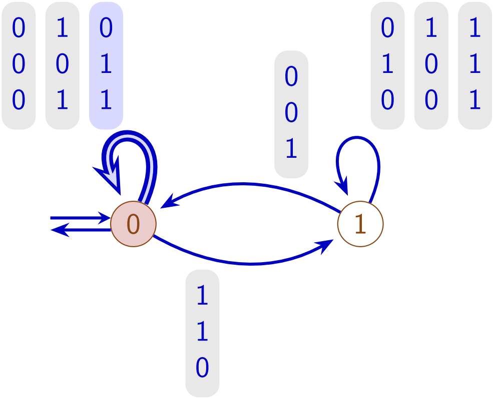

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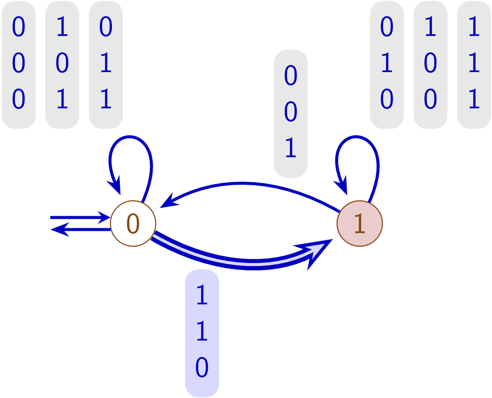

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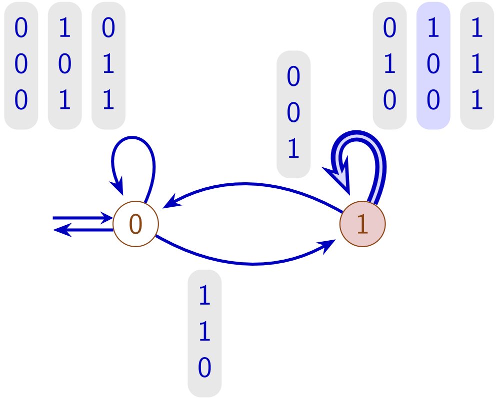

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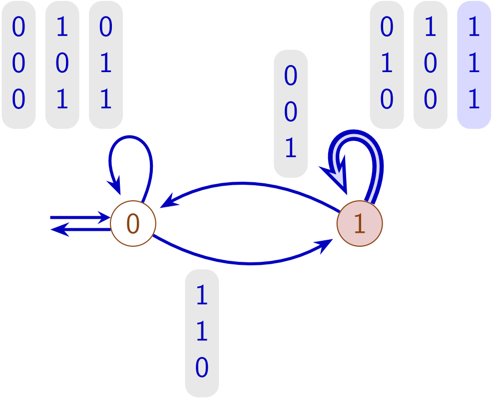

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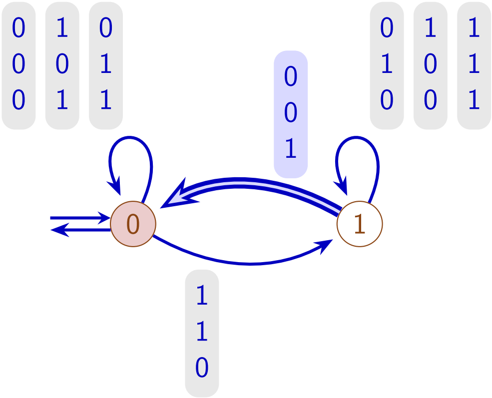

Addition is just a *Digit-conversion*

Addition

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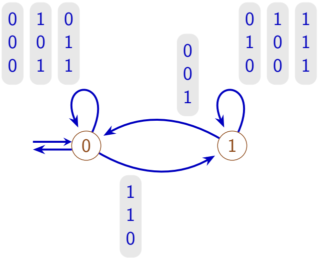

Figure: Additioner

Addition is just a *Digit-conversion*

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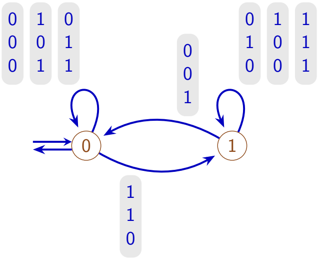

Figure: Additioner

Digit-conversion $\{0,1,2\} \to \{0,1\}$

0	2	1	2	1	0
1	1	0	0	1	0

Same value

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carry-less addition

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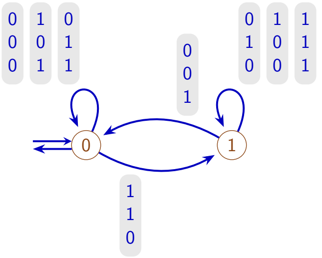

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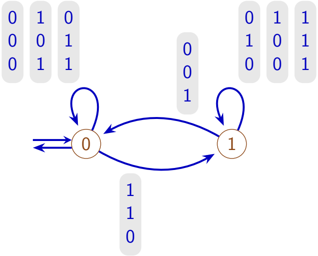
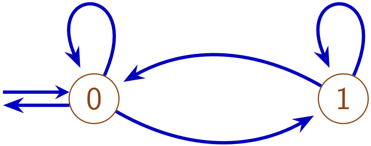


Figure: Additioner



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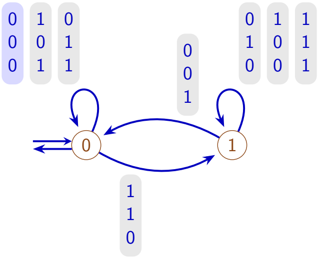

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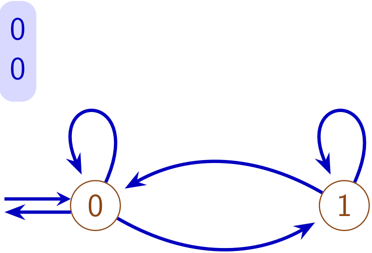

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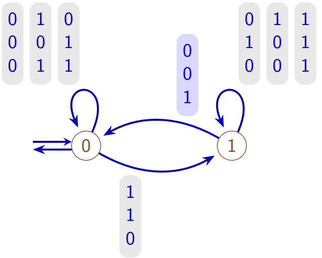

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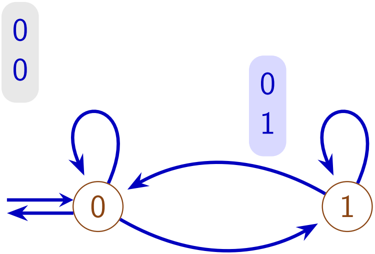

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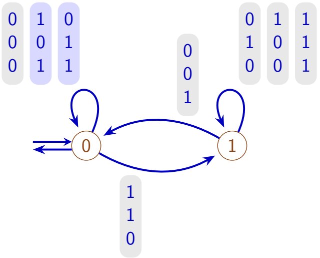

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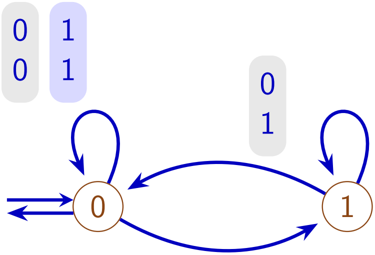

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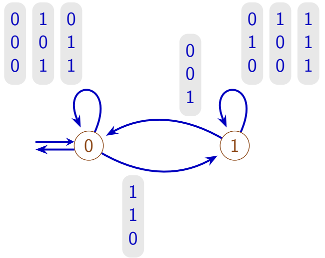

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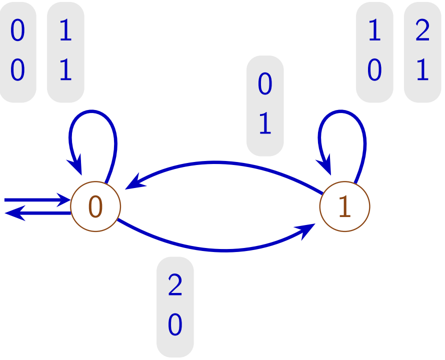

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One strong theorem based the additioner

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Büchi-Bruyère Theorem [Büchi'60][Bruyère'85][Charlier'18]

R : a relation in $\mathbb{N}^d$.
R is realised by a d-tape automaton in base p
$\iff$ R is definable by a formula in $FO[\mathbb{N}, +, V_p]$ †

†: $FO[\mathbb{N}, +, V_p]$ is the first-order logic with functions $+$ and V_p.
V_p is the function $n \mapsto 2^k$, the greater power of 2 such that $\frac{n}{2^k} \in \mathbb{N}$.

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⇒ Many properties are decidable for automatic sequences (periodicity, squarefreeness, etc.)

2 Journey from evaluator to additioner

Base β

$\beta > 1$: a real number

Evaluation in base β

Before the radix point

$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \sum_{i=0}^{k} a_i \beta^i \tag{4}$$

After the radix point

$$\mathrm{Val}(\bullet a_1 \cdots a_k \cdots) = \sum_{i=0}^{\infty} a_i \beta^{-i} \tag{5}$$

Base β

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$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \sum_{i=0}^{k} a_i \beta^i \tag{4}$$

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In this talk, we do not care about representation (Greedy, normalisation, ...)

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- Edges: $s \xrightarrow{a} t \iff s\beta + a = t$

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$$\mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = \beta \times \mathrm{Val}(a_k \cdots a_1 \bullet) + a_0$$

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$$\overbrace{\mathrm{Val}(a_k \cdots a_1 a_0 \bullet)}^{x=} = \beta \times \overbrace{\mathrm{Val}(a_k \cdots a_1 \bullet)}^{y=} + a_0$$

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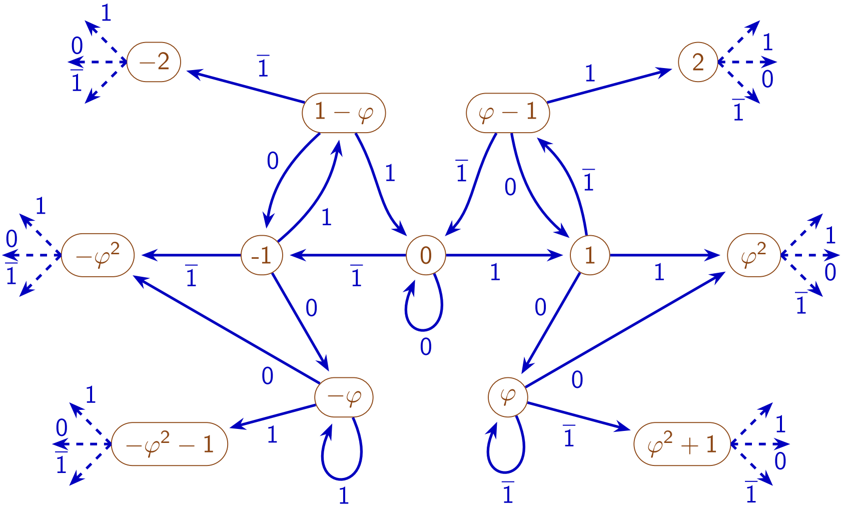
Lemma

For each $a_k \cdots a_1 a_0$,

$$0 \xrightarrow{a_k} \cdots \xrightarrow{a_1} \xrightarrow{a_0} s \qquad \text{where } s = \mathsf{Val}(a_k \cdots a_1 a_0 \bullet) \tag{6}$$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



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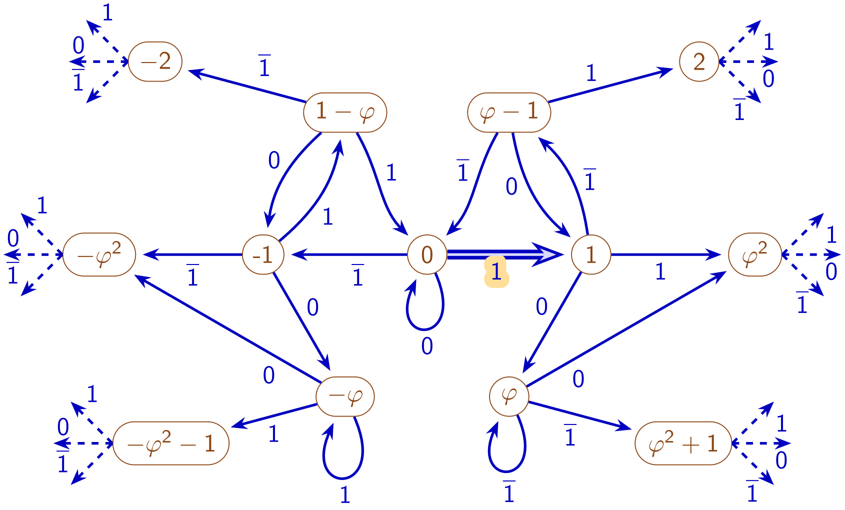
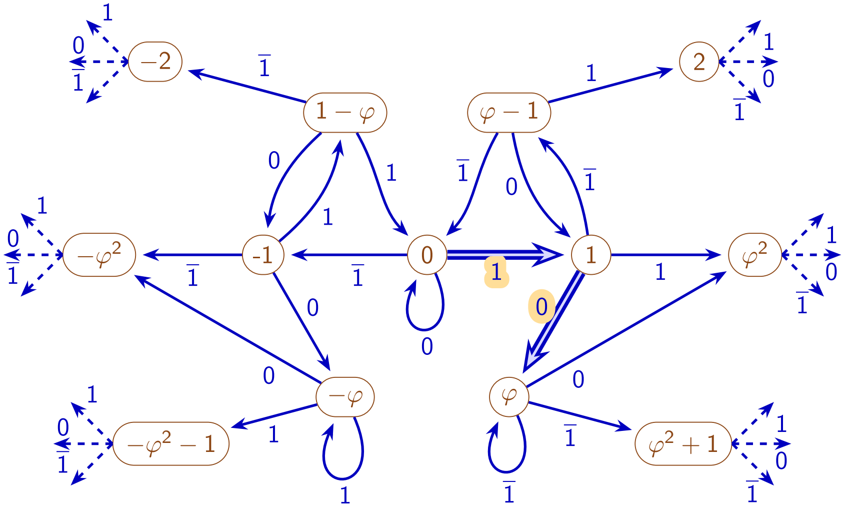


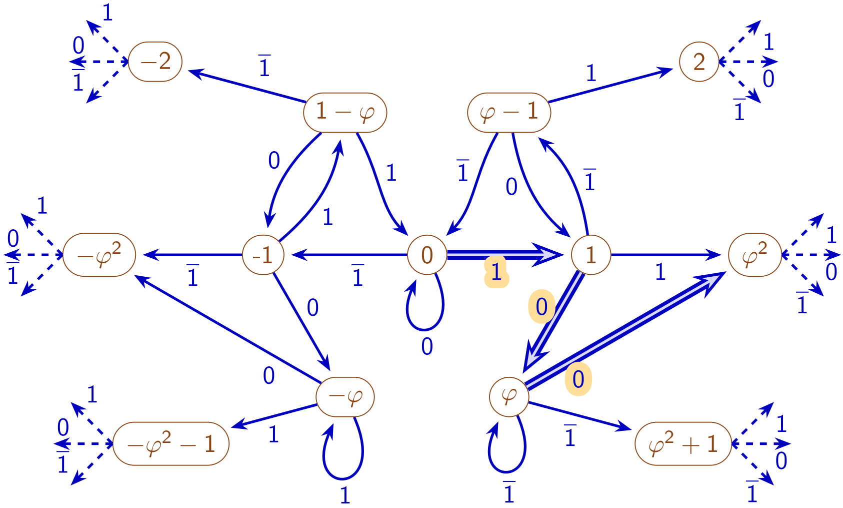
$\mathsf{Val}(1\bullet) = 1$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



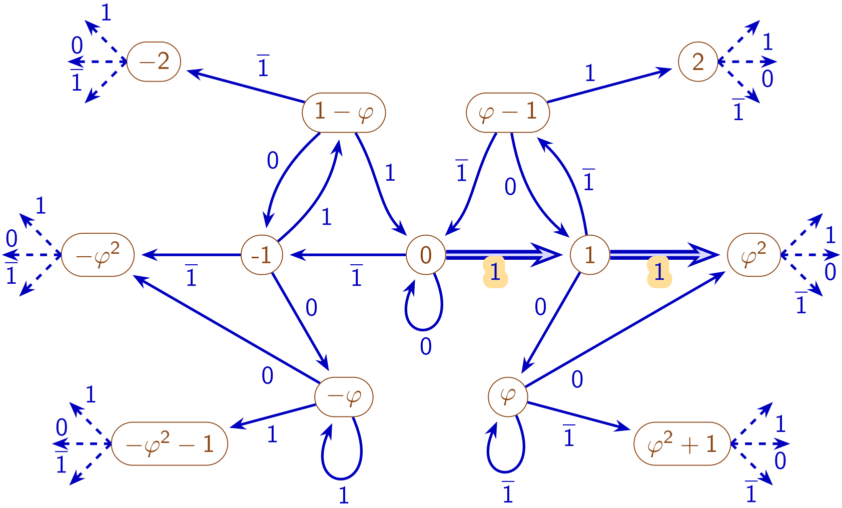
$$\mathsf{Val}(10_\bullet) = \varphi$$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



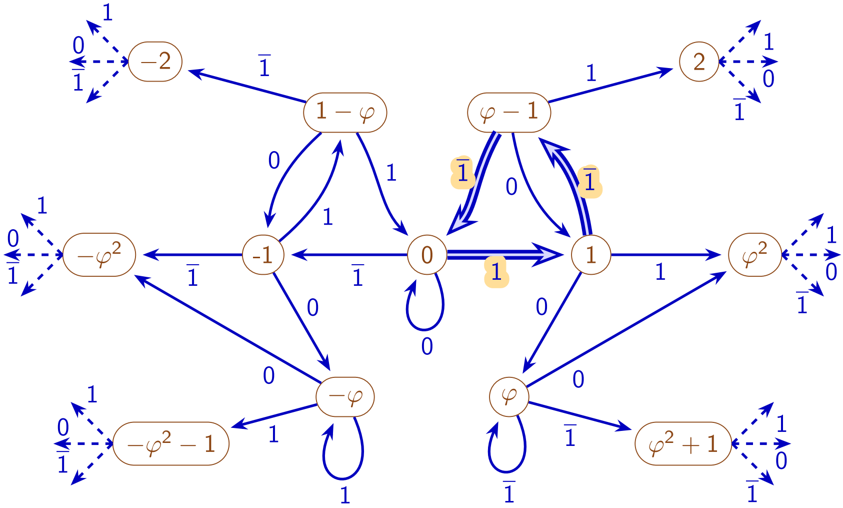
$\mathsf{Val}(100\bullet) = \varphi^2$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



$$\mathrm{Val}(11\bullet) = \varphi + 1 = \varphi^2$$

$\mathcal{E}_A$ with $A = \{\overline{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



$$\mathrm{Val}(1\overline{1}\overline{1}\bullet) = \varphi^2 - \varphi - 1 = 0$$

The zero-automaton $\mathcal{Z}_A$

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$\mathcal{Z}_A$ is defined as $\mathcal{E}_A$ except:

- $\mathcal{Z}_A$ is a Büchi automaton
- States: vertices of $\mathcal{E}_A$ in $\left[-\frac{d}{\beta-1}, \frac{d}{\beta-1}\right]$, where $d = \max_{a \in A}\{|a|\}$
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- All states are final

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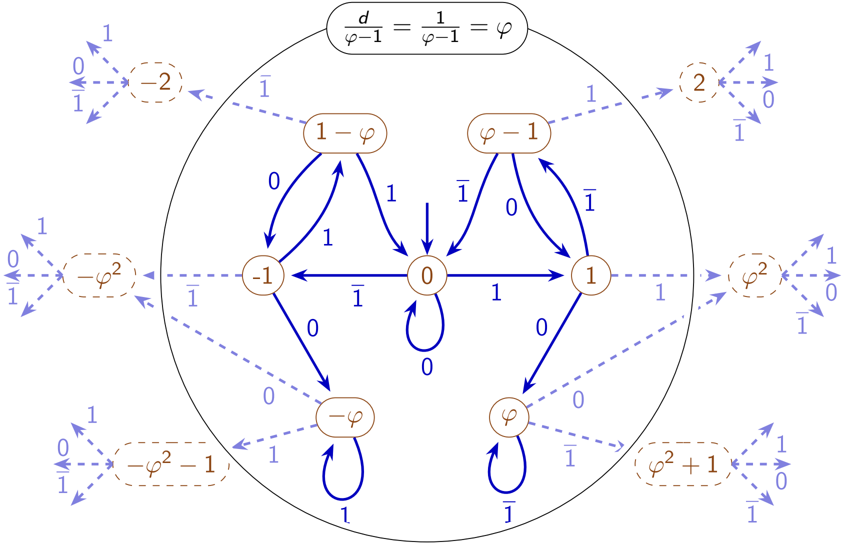
Theorem [Frougny'92]

$(a_0 a_1 \cdots a_k \cdots) \in A^{\mathbb{N}}$ is accepted by $\mathcal{Z}_A$

$$\iff \mathrm{Val}(\bullet a_0 a_1 \cdots a_k \cdots) = 0$$

$\mathcal{E}_A \to \mathcal{Z}_A$ with $A = \{\bar{1}, 0, 1\}$ and $\beta = \varphi = \frac{1+\sqrt{5}}{2}$



Zero-automaton Theorem

The following are equivalent.

1. β is a Pisot number.
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1 $\Rightarrow$ 2 in [Frougny'92]

2 $\Rightarrow$ 1 in [Berend-Frougny'94]

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Corollary

$a_k \cdots a_1 a_0$ accepted by $\mathcal{F}_A \iff \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = 0$

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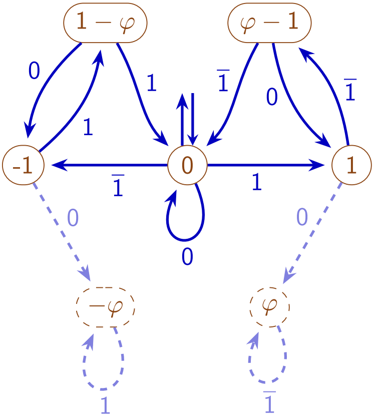

Converter $\mathcal{C}_{D_1,D_2}$

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Behavior we want

Accept pairs $\begin{matrix} a_k \cdots a_1 a_0 & \in D_1 \\ b_k \cdots b_1 b_0 & \in D_2 \end{matrix}$ such that

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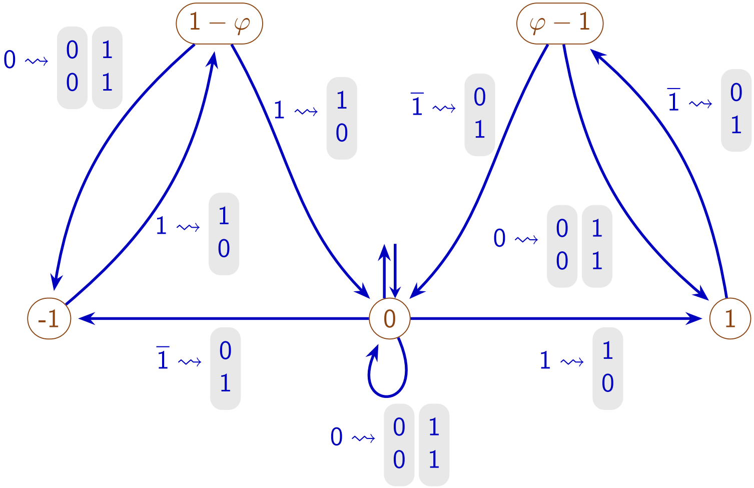
&\iff (a_k - b_k) \cdots (a_1 - b_1)(a_0 - b_0) \text{ is accepted by } \mathcal{F}_A
\end{aligned}$$

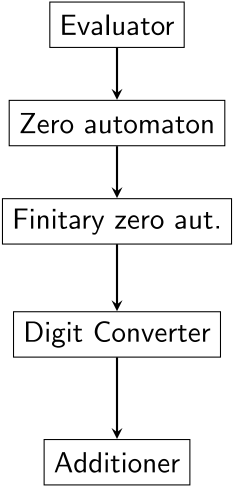
where $A = \{(d_1 - d_2) \mid d_1 \in D_1,\ d_2 \in D_2\}$

$\mathcal{F}_A \to \mathcal{C}_{D_1,D_2}$ with $A = \{\bar{1}, 0, 1\}$ and $D_1 = D_2 = \{0, 1\}$



$$a \in \{\bar{1}, 0, 1\} \rightsquigarrow \left\{ \begin{matrix} d_1 \in \{0,1\} \\ d_2 \in \{0,1\} \end{matrix} \;\middle|\; a = (d_1 - d_2) \right\}$$

Recap: from evaluator to additioner



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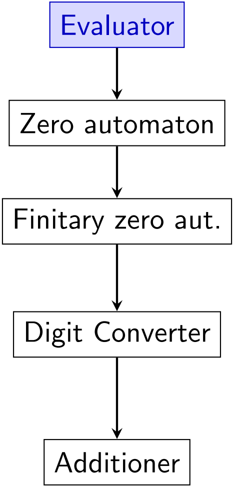

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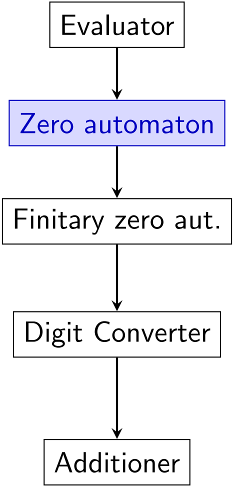

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Zero automaton	$a_0 a_1 \cdots a_k \cdots$	accepted iff $\mathsf{Val}(\bullet a_0 a_1 \cdots a_k \cdots) = 0$

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A statement similar to the Büchi-Bruyère Theorem

$U = (U_k)_{k \in \mathbb{N}}$ a Pisot linear system

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Theorem [Bruyère-Hansel'97]

R : a relation in $\mathbb{N}^d$.

R is realised by a d-tape automaton in U

$\iff$ R is definable by a formula in $FO[\mathbb{N}, +, V_U]$ †

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$\Rightarrow$ Many properties are decidable for U-automatic sequences (periodicity, squarefreeness, etc.)

3 Foreshadowing: Rational base

The case where $\beta = \frac{p}{q} \in \mathbb{Q}$

Evaluation in base $\frac{p}{q}$

Before the radix point

$$\mathrm{Val}(a_n \cdots a_1 a_0 \bullet) = \sum_{i=0}^{n} a_i \left(\frac{p}{q}\right)^i \qquad (7)$$

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Theorem [Akiyama-Frougny-Sakarovitch'08]

$\forall n \in \mathbb{N}$,

$$\exists a_k \cdots a_1 \in \{0, \ldots, p-1\} \qquad \mathrm{Val}(a_k \cdots a_1 a_0 \bullet) = n \tag{8}$$

Base $\frac{p}{q}$ features an additioner

Theorem [Akiyama-Frougny-Sakarovitch'08]

Addition is realised by a finite automaton in base $\frac{p}{q}$.

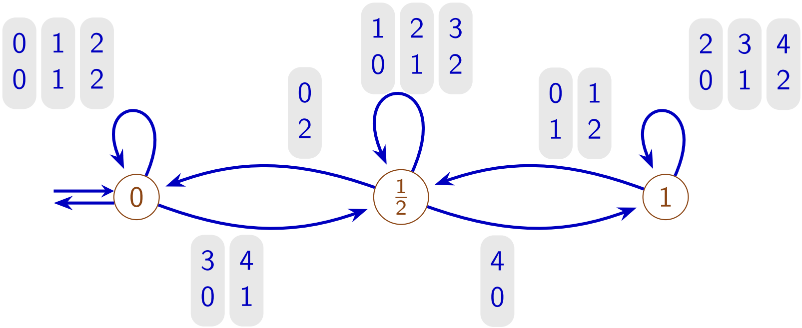

Figure: Converter $\{0,1,2,3,4\} \to \{0,1,2\}$ in base $\frac{3}{2}$

Wait? Rational numbers are **not** Pisot numbers

β is a Pisot number iff

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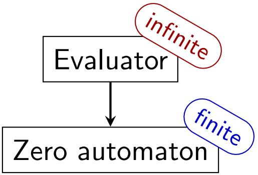
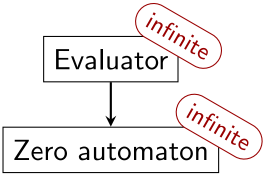
Explanation of the non-contradiction

If β is Pisot



If β is **not** Pisot



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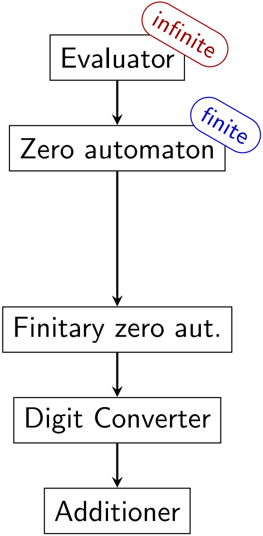

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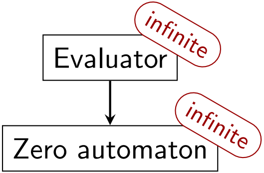

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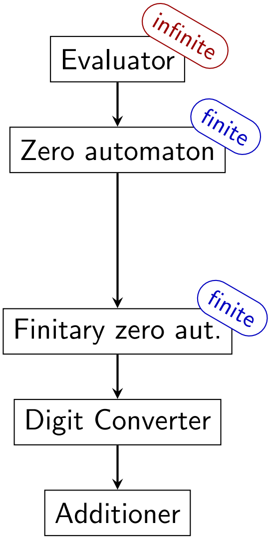

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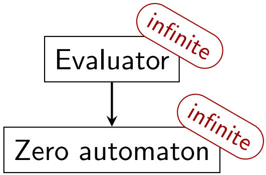

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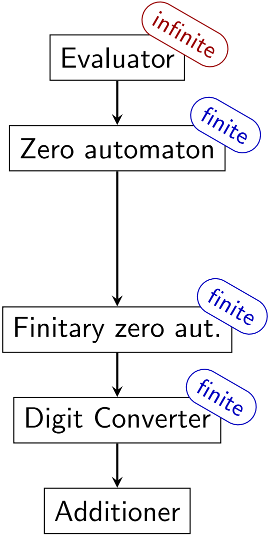

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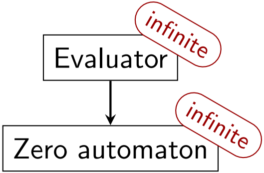

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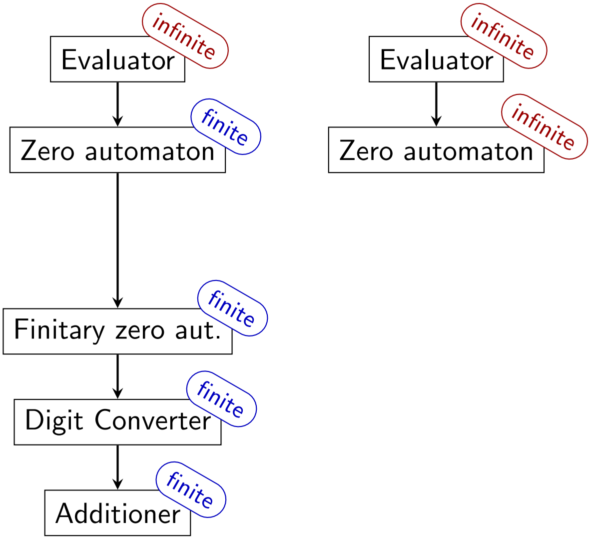

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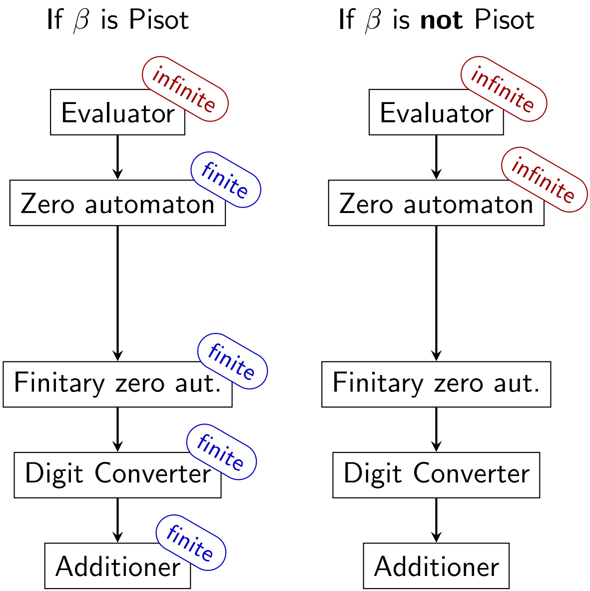

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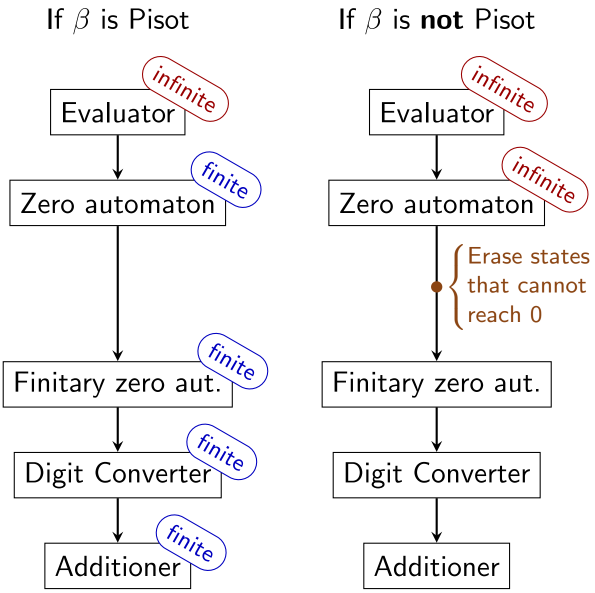

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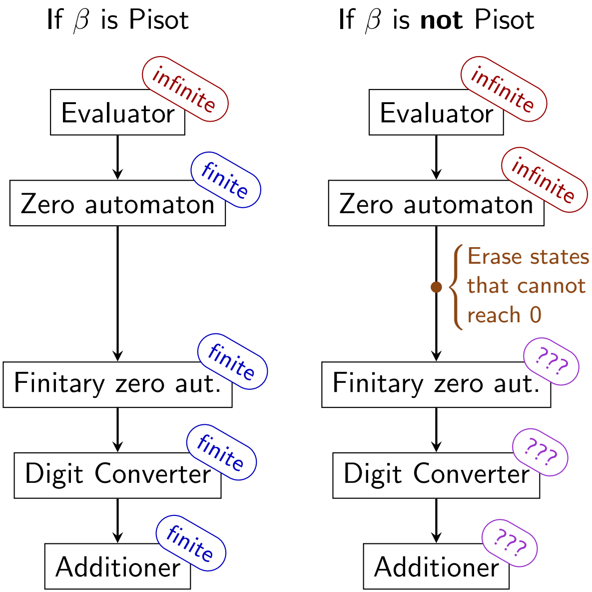
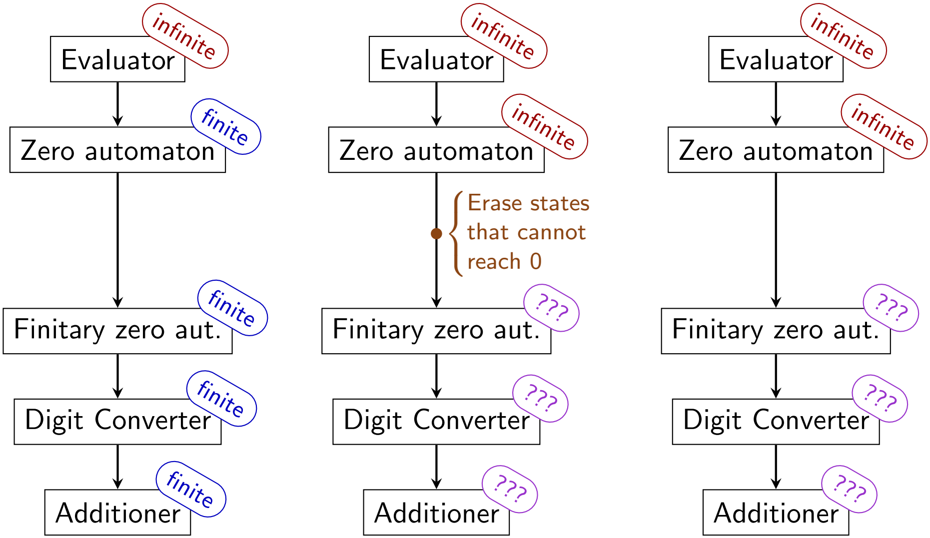


Explanation of the non-contradiction

If β is Pisot — If β is **not** Pisot — If $\beta = \frac{p}{q}$



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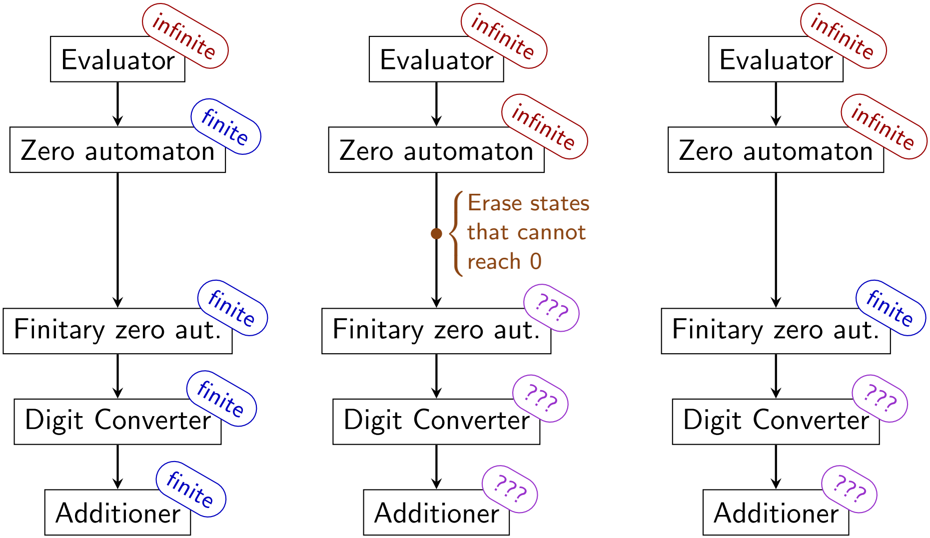

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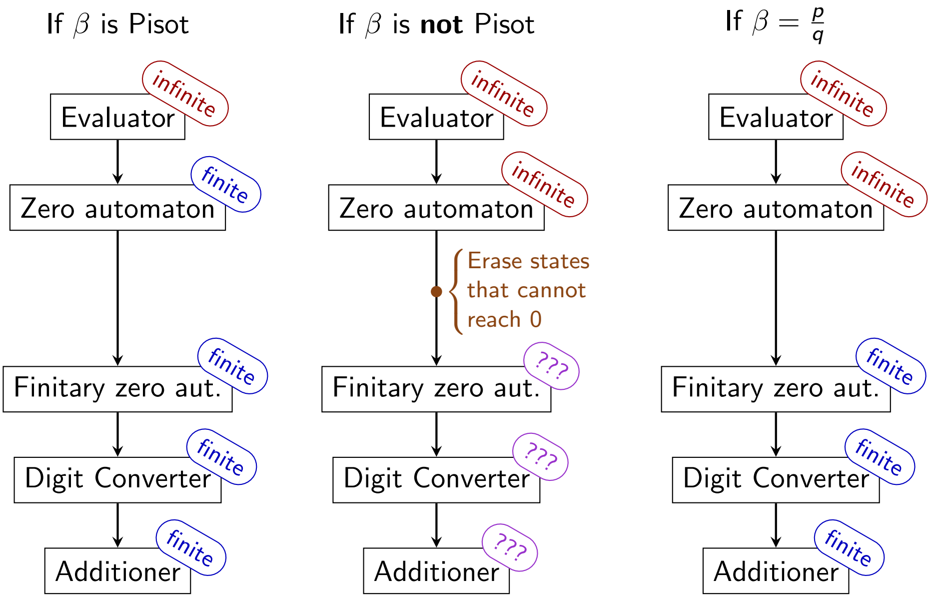

$N_{\frac{p}{q}} = \mathrm{Val}(A^*)$ with $A = \{0, \ldots, p-1\}$

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$V_{\frac{p}{q}}$ is the function $n \mapsto \left(\frac{p}{q}\right)^k$, the greater power of $\frac{p}{q}$ such that $n\left(\frac{p}{q}\right)^{-k} \in N_{\frac{p}{q}}$.

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What about $\frac{p}{q}$-automatic sequences?

$\rightarrow$ Mostly open

Recap: from evaluator to additioner

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