k-Regular Sequences: Computations and Asymptotic Analysis

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Binary Sum-of-Digits Function

Example

$$s(26) =$$

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$$s(2^j n + r) = s(n) + s(r), \quad 0 \le r < 2^j$$

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Rewriting as Linear Combinations

$s(2^j n + r) = 1 \cdot s(n) + c_{jr} \cdot 1$ for every $j \ge 0$, $0 \le r < 2^j$

k-regular Sequences

explicitly:

- there exist sequences $f_1(n), \dots f_s(n)$ such that
- for all $j \geq 0$, $0 \leq r < k^j$
- there exist $c_1, \dots, c_s$
- with
$$f(k^j n + r) = c_1 f_1(n) + \cdots + c_s f_s(n)$$

k-regular Sequences

k-regular Sequence $f(n)$

k-kernel $\left\{f(k^j n + r) \mid j \geq 0,\ 0 \leq r < k^j\right\}$
is contained in
finitely generated module

$\langle \dots \rangle$

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k-linear Representation

binary sum of digits $s(n)$:

- recurrence relations

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- vector-valued sequence

  set $v(n) = (s(n), 1)^T$

  even $v(2n) = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n)$

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k-regular Sequence $f(n)$

- square matrices $M_0, \ldots, M_{k-1}$
- vectors u and w
- k-linear representation

$$f(n) = u^T M_{n_0} M_{n_1} \ldots M_{n_{\ell-1}} w$$

with standard k-ary expansion

$$n = (n_{\ell-1} \ldots n_1 n_0)_k$$

Some k-regular Sequences

- largest power of k less than or equal to n
- k-ary sum of digits
- redundant systems:
  number of representations in base k

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- k-recursive sequences
  - Stern's Diatomic Sequence
  - Generalized Pascal's Triangle
  - Number of Unbordered Factors in the Thue–Morse Sequence
- completely k-additive functions

Why should we care?

- natural generalization of automatic sequences to infinite alphabet
  (automatic $\iff$ only finitely many values)
- several alternative characterizations
- recognizable series / non-commutative rational series
  (Berstel–Reutenauer 2011)

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- rich arithmetic structure ($\rightsquigarrow$ SageMath):
  - $+$. scalar multiplication, convolution
    ($\rightsquigarrow$ module $+$ ring $=$ algebra)
  - shifts and linear subsequences
  - pointwise multiplication, partial sums, modulo
  - …and many more
- computability

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- computability
- growth $O(n^c)$ for some constant c

Pascal's Rhombus

				1				
			1	1	1			
		1	2	4	2	1		
	1	3	8	9	8	3	1	
1	4	13	22	29	22	13	4	1

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Pascal's Rhombus

0

0 0 1

1 1 1

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1 3 8 9 8 3 1

1 4 13 22 29 22 13 4 1

Recurrence

$$r_{i,j} = r_{i-2,j} + r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1}$$

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$$r_{i,j} = \begin{matrix} r_{i-2,j} + \\ r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} \end{matrix}$$

Pascal's Rhombus

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1 1 1 **0**

1 2 4 **2** **1** **0**

1 3 8 9 8 **3** 1

1 4 13 22 29 22 13 4 1

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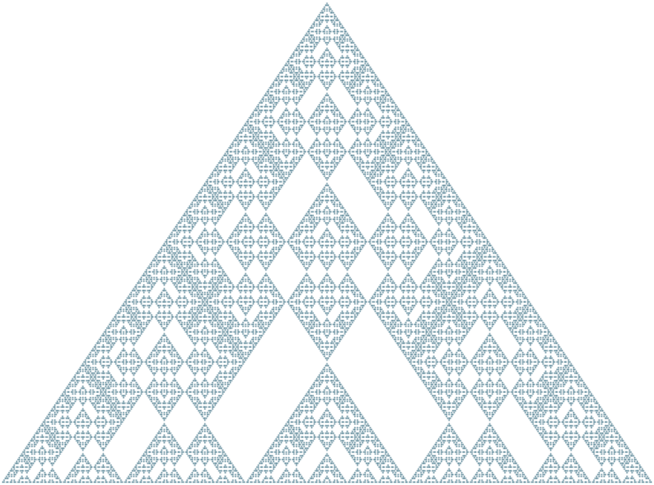
Question

Where are the odd entries?

Pascal's Rhombus Modulo 2



Pascal's Rhombus: Recurrence Relations

- Pascal's rhombus $\mathfrak{R}$ modulo 2
- split into odd/even row and column indices:
  - $\mathfrak{X}$ (even rows and columns)
    $\rightsquigarrow$ odd entries x_n
  - $\mathfrak{Y}$, $\mathfrak{Z}$
    $\rightsquigarrow$ odd entries y_n, z_n
  - $\mathfrak{U}$ (odd rows and columns)
    $\rightsquigarrow$ odd entries u_n
- $\mathfrak{A} = \mathfrak{R}$, $\mathfrak{U} = 0$

Recurrences (Goldwasser–Klostermayer–Mays–Trapp 1999)

$$x_{2n} = x_n + z_n \qquad x_{2n+1} = y_{n+1}$$
$$y_{2n} = x_{n-1} + z_n \qquad y_{2n+1} = x_{n+1} + z_n$$
$$z_{2n} = 2x_n \qquad z_{2n+1} = 2y_{n+1}$$

Pascal's Rhombus: 2-regular Sequences

- coefficient vector
  - $v_n = (x_n, x_{n+1}, y_{n+1}, z_n, z_{n+1})^T$
  - with $v_0 = (0, 1, 1, 0, 2)^T$
- rewrite recurrence

$$v_{2n} = M_0 v_n$$

and

$$v_{2n+1} = M_1 v_n$$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

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$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

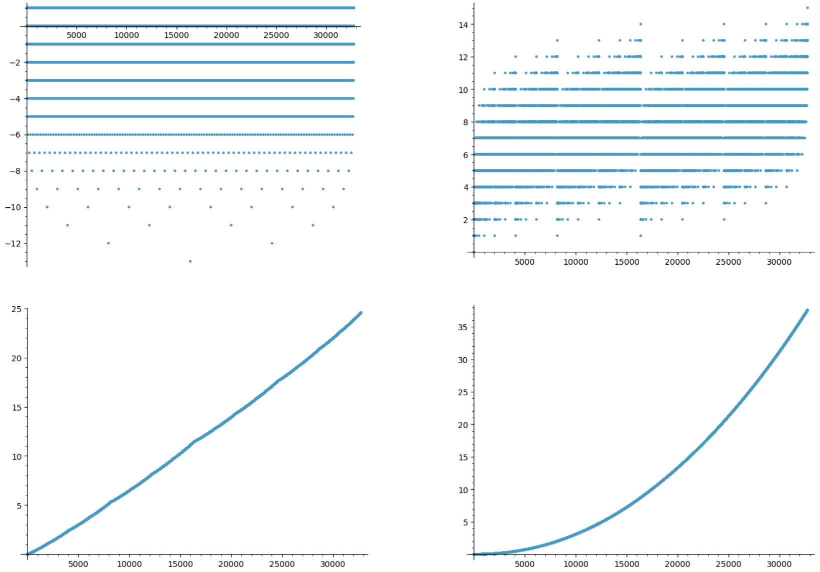
2-linear Representation

- sequence
  - binary expansion $n = (\delta_{\ell-1} \dots \delta_1 \delta_0)_2$
  - linear representation

$$v_n = M_{\delta_0} M_{\delta_1} \dots M_{\delta_{\ell-1}} v_0$$

  - $\rightsquigarrow$ sequences x_n, y_n, z_n are 2-regular

Some k-regular Sequences



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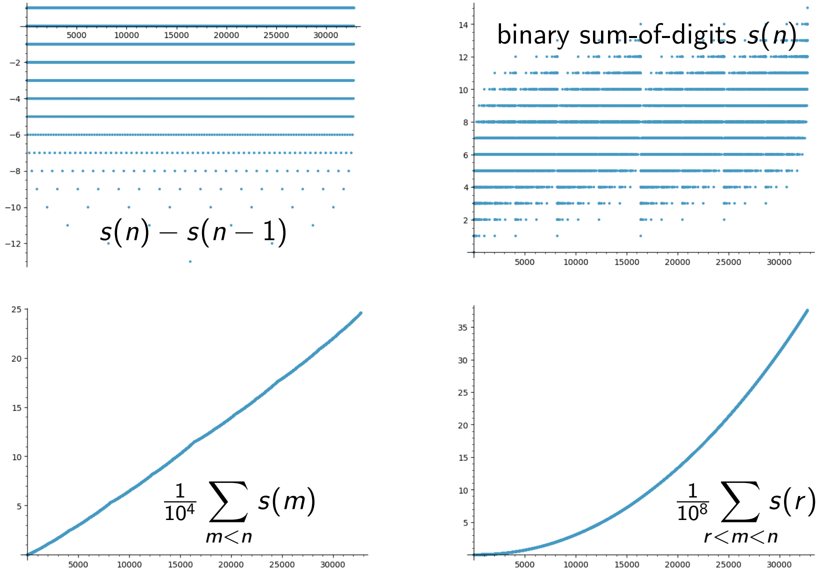

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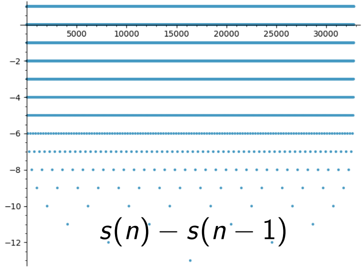

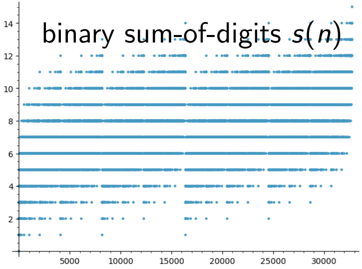

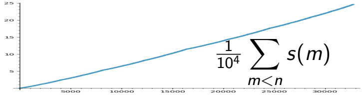

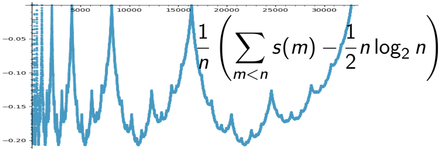

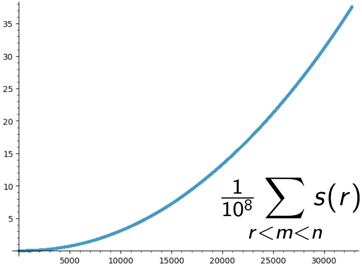

Asymptotics & Fluctuations

- properties of particular sequences:
  - binary sum of digits
    (Delange 1975)
  - optimal digit expansions
    (Grabner–Heuberger–Prodinger 2005, Grabner–Heuberger 2006)
  - subword occurrences
    (Leroy–Rigo–Stipulanti 2016)
  - . . . plenty more
- classes of sequences:
  - divide-and-conquer algorithms
    (Drmota–Szpankowski 2013, Hwang–Janson–Tsai 2017)
  - output sums of transducers
    (Heuberger–Kropf–Prodinger 2015)
  - non-commutative rational series
    (Dumas–Lipmaa–Wallén 2007)
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- fluctuations, functional equations:
  - periodicity phenomena
    (Flajolet–Grabner–Kirschenhofer–Prodinger–Tichy 1994)
  - automatic sequences
    (Allouche–Mendès France–Peyrière 2000)
- k-regular sequences (via dilation equations)
  (Dumas 2013, Dumas 2014)

Asymptotics of Partial Sums

- k-regular sequence $f(m)$
  - matrices $M_0, \ldots, M_{k-1} \in \mathbb{C}^{d\times d}$
  - sequence $(f(m))_{m\geq 0}$ of matrices with $f(km + r) = M_r f(m)$, $f(0) = I$
- interested in asymptotic behaviour of $F(n) = \sum_{0\leq m<n} f(m)$

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- interested in asymptotic behaviour of $F(n) = \sum_{0\leq m<n} f(m)$

Theorem (Dumas 2013)

(condensed statement out of the formulation in Heuberger–K–Prodinger 2016)

$$F(n) = \sum_{\substack{\lambda\in\sigma(M_0+\cdots+M_{k-1})\\ |\lambda|>\rho}} n^{\log_k \lambda} \sum_{0\leq\ell\leq m(\lambda)} (\log_k n)^\ell \Phi_{\lambda,\ell}(\{\log_k n\}) + O(n^{\log_k R}(\log n)^{\widehat{m}})$$

- 1-*periodic (Hölder) continuous functions* $\Phi_{\lambda,\ell}$

Asymptotics of Partial Sums

- k-regular sequence $f(n)$
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Theorem (Heuberger–K–Prodinger 2018, Heuberger–K 2020)

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- *1-periodic (Hölder) continuous functions* $\Phi_{\lambda\ell}$
- *functional equation*

$$\Big(I - \frac{1}{k^s}(M_0+\cdots+M_{k-1})\Big)\mathcal{V}(s) = \sum_{n=1}^{k-1} \frac{v(n)}{n^s} + \frac{1}{k^s}\sum_{r=0}^{k-1} M_r \sum_{\ell\ge1}\binom{-s}{\ell}\Big(\frac{r}{k}\Big)^\ell \mathcal{V}(s+\ell)$$

- *meromorphic continuation on the half plane* $\Re s > \log_k R$
- *Fourier series* $\Phi_{\lambda\ell}(u) = \sum_{h\in\mathbb{Z}} \varphi_{\lambda\ell h} \exp(2\ell\pi i u)$

$$\varphi_{\lambda\ell h} = \frac{(\log k)^\ell}{\ell!} \operatorname{Res}\left(\frac{\big(f(0)+\mathcal{F}(s)\big)\big(s-\log_k\lambda-\frac{2h\pi i}{\log k}\big)^\ell}{s}, s = \log_k\lambda + \frac{2h\pi i}{\log k}\right)$$

Binary Sum-of-Digits Function: Analysis

- eigenvalues etc.
  - $C = M_0 + M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$
  - C has eigenvalue $\lambda = 2$ with multiplicity 2
  - joint spectral radius 1
  - $\|M_{r_1} \cdots M_{r_\ell}\| = O\big(R^\ell\big)$ for any $R > 1$
- $\rightsquigarrow$ analysis of summatory function:
  - $S(N) = N(\log_2 N)\, \Phi_{21}(\{\log_2 N\}) + N\, \Phi_{20}(\{\log_2 N\})$
  - 1-periodic continuous functions Φ_{21} and Φ_{20}
  - $\Phi_{21}(u) = \frac{1}{2}$ via functional equation
  - no error term
- recovering:

Summatory Binary Sum-of-Digits (Delange 1975)

$$S(N) = \sum_{n<N} s(n) = \tfrac{1}{2} N \log_2 N + N\, \Phi_{20}(\{\log_2 N\})$$

- explicit Fourier coefficients of $\Phi_{20}(u)$

Pascal's Rhombus: The Odds

- $x_n =$ number of odd entries in line n in Pascal's rhombus

✓

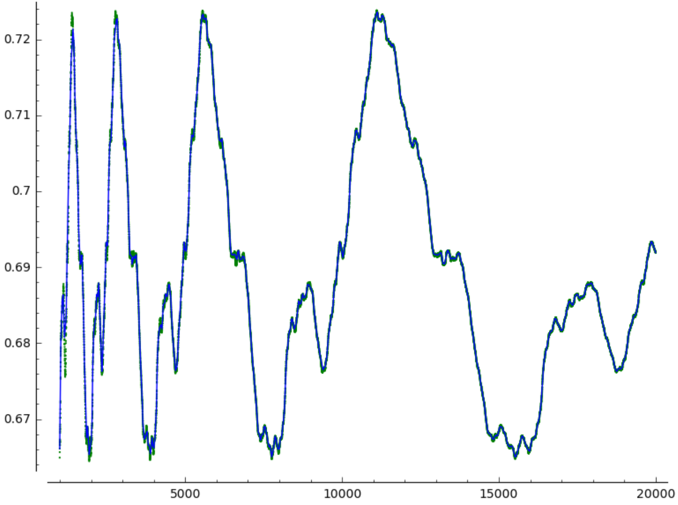
Theorem (Heuberger–K–Prodinger 2018)

number of odd entries in the first N lines

$$X_N = \sum_{n \leq N} x_n = N^\kappa \, \Phi(\log_2 N) + O(N \log_2 N)$$

- $\kappa = 2 - \log_2(\sqrt{17} - 3) = 1.8325063835804\ldots$
- *continuous and 1-periodic function* $\Phi(u)$
- *Fourier coefficients* ✓

Pascal's Rhombus: The Fluctuation



fluctuation $\Phi(\log_2 n)$ vs. a_n/n^κ

Dirichlet Series for Pascal's Rhombus

- numbers x_n, y_n, z_n
  of odd entries
  in Pascal's rhombus
  in line n

- Dirichlet series
  $X(s) = \sum_{n\geq 1} \frac{x_n}{n^s}$
  $Y(s) = \sum_{n\geq 1} \frac{y_n}{n^s}$
  $Z(s) = \sum_{n\geq 1} \frac{z_n}{n^s}$

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Functional Equation System of Dirichlet Series

- recurrence relations $\rightsquigarrow$

$$\begin{pmatrix} X(s) \\ Y(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} 2^{-s} & 2^{-s} & 2^{-s} \\ 2^{1-s} & 0 & 2^{1-s} \\ 2^{1-s} & 2^{1-s} & 0 \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \\ Z(s) \end{pmatrix} + \begin{pmatrix} J \\ UN \\ K \end{pmatrix}$$

- $JUNK$ contains
  - $\sum_{\ell\geq 1} \ldots X(s+\ell)$
  - $\sum_{\ell\geq 1} \ldots Y(s+\ell)$
  - $\sum_{\ell\geq 1} \ldots Z(s+\ell)$

Mellin–Perron Summation Formula of Order 0

- Mellin transform
  - $H(s) = \int_0^\infty h(x)x^{s-1}\,\mathrm{d}x$
  - inverse transform
    $h(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\vartheta} H(s)x^{-s}\,\mathrm{d}s$

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  - sequence $(d_n)_{n\geq 1}$
  - $D(s) = \sum_{n\geq 1} \frac{d_n}{m^s}$

The Formula

$$D_N - \frac{d_N}{2} = \sum_{n=1}^{N-1} d_n + \frac{d_N}{2} = \sum_{n\geq 1} d_n \left[0 \leq \frac{n}{N} < 1\right] + \frac{d_N}{2}$$

$$= \sum_{n\geq 1} d_n \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\vartheta} \left(\frac{n}{N}\right)^{-s} \frac{\mathrm{d}s}{s}$$

$$= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\vartheta} \frac{1}{s} \left(\sum_{n\geq 1} \frac{d_n}{n^s}\right) N^s\,\mathrm{d}s = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\vartheta} D(s) \frac{N^s}{s}\,\mathrm{d}s$$

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  - ...seems to give asymptotic behaviour
    ...and possibly Fourier coefficients
  - ...but convergence issues

Decidability

Decision Problem

problem with a yes/no answer

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Recursively Solvable, Solvable, Decidable

for decision problem
there exists an algorithm (or Turing machine)
that unerringly solves it on all inputs

Is Prime?

"Given a natural number,
is it prime?"

...decidable?

Integer Roots of Polynomials

"Given a univariate polynomial
with integer coefficients,
does it have an integer root?"

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Rational Roots of Polynomials

"Given a univariate polynomial
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Roots of Multivariate Polynomials

"Given a multivariate polynomial p with integer coefficients,
do there exist natural numbers x_1, x_2, ..., x_t
such that $p(x_1, \dots, x_t) = 0$?"

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Hilbert's tenth problem; variant (MRDP 1949–1970)

"Given a multivariate polynomial p with integer coefficients,
do there exist natural numbers x_1, x_2, . . . , x_t
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. . . recursively unsolvable

Equality of k-regular Sequences

Theorem (K–Shallit 2020)

"Given two k-regular sequences $(f(n))_{n\geq 0}$ and $(g(n))_{n\geq 0}$ over $\mathbb{Q}$, does $f(n) = g(n)$ for all n hold?"

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- Proof:
  - compute linear representation of $f(n) - g(n)$
  - apply minimization algorithm
  - rank 0 iff $f(n) - g(n) = 0$ for all n

Zero Terms

Theorem (Allouche–Shallit 1992)

"Given a k-regular sequence over $\mathbb{N}_0$, does it have a zero term?"

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- Proof:
  - multivariate polynomial p in t variables over $\mathbb{Z}$
  - choose $r \in \mathbb{N}$ large enough
  - $f(n) = p(|z|_1, |z|_2, \dots, |z|_t)$
    - z equals k^r-representation of n
    - $|z|_d$ is number of occurrences of letter d in z
  - $(f(n))_{n \geq 0}$ is k^r-regular and consequently k-regular

More (Un-)Decidability Results

- recursively unsolvable:
  - Image of sequence equals $\mathbb{N}$? ...equals $\mathbb{Z}$?
  - Images of two sequences coincide?
  - Sequences takes same value twice?
  - Sequence contains a square?
  - Sequence contains a palindrom?
  - Preimages can be recognized by a deterministic finite automaton?

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  "no nontrivial property is decidable"

  (Honkala 2021)
- recursively solvable for k-automatic sequences:
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  - Sequence contains a palindrom?
  - . . .

  (e.g. Charlier–Rampersad–Shallit 2012)

Undecidability of the Growth

Theorem (K–Shallit 2020)

"Given a k-regular sequence $(f(n))_{n\geq 0}$ *over* $\mathbb{Q}$*, is* $f(n)$ *in* $O(n^\sigma (\log n)^\ell)$*?"*

... recursively unsolvable

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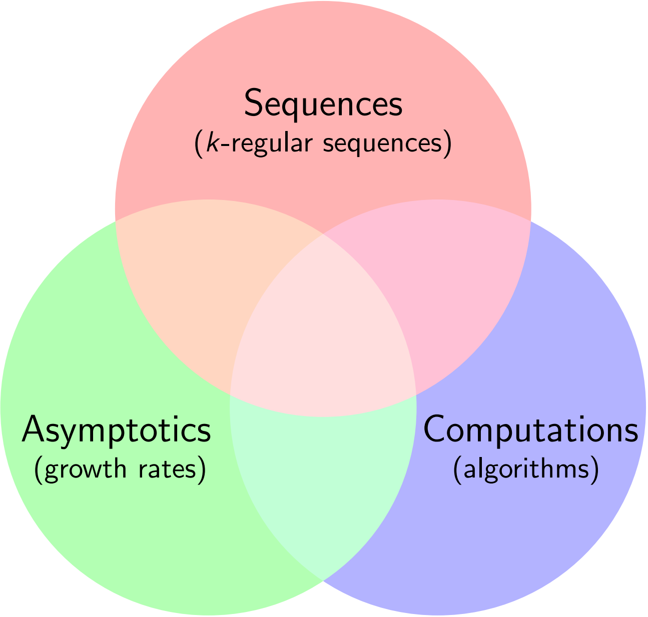
n^2


Sequences
(k-regular sequences)
Asymptotics
(growth rates)
Computations
(algorithms)