Lattice-based number systems with the same radix

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Definitions

Let $R$ be a ring, $\beta \in R$ (radix) and $\mathcal{A} \ni 0$ a finite subset of $R$ (alphabet). A $(\beta, \mathcal{A})$-representation of $x \in R$ is

$$x = \sum_{i=0}^{N} \beta^i a_i, \quad \text{where } N \in \mathbb{N}_0, a_i \in \mathcal{A}, a_N \neq 0.$$ 

Definition

We call $(\beta, \mathcal{A})$ a number system (GNS) on $R$ if every nonzero $x \in R$ has a unique $(\beta, \mathcal{A})$-representation.

Which of the following are GNSs (in $\mathbb{Z}$ or $\mathbb{Z}[i]$)?

$(10, \{0, \ldots, 9\}); (2, \{0, 1\}); (2, \{-1, 0, 1\}); (-2, \{0, 1\}); (3, \{-1, 0, 1\}); (10, \{-5, \ldots, 5\}); (1 + i, \{0, 1\}); (-1 + i, \{0, 1\}).$

Answer: The negabinary, the balanced ternary and the Penney number system.
Number systems in rings

Definitions

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Which of the following are GNSs (in $\mathbb{Z}$ or $\mathbb{Z}[i]$)?

- $(10, \{0, \ldots, 9\})$
- $(2, \{0, 1\})$
- $(2, \{-1, 0, 1\})$
- $(-2, \{0, 1\})$
- $(3, \{-1, 0, 1\})$
- $(10, \{-5, \ldots, 5\})$
- $(1 + i, \{0, 1\})$
- $(-1 + i, \{0, 1\})$

Answer: The negabinary, the balanced ternary and the Penney number system.
Penney, 1965: \((-1 + i, \{0, 1\})\) is a GNS on \(\mathbb{Z}[i]\).

(For example, \(-1 = \beta^4 + \beta^3 + \beta^2 + 1\) for \(\beta = -1 + i\).)

Negative powers \(\sum_{j=-\infty}^{N} \beta^j a_j\) allow to represent all of \(\mathbb{C}\):
Let \((\beta, A)\) be a GNS in a number field order \((\mathcal{O} \subset \mathcal{O}_K\) for a number field \(K\)). Then:

1. \(A\) is a FRS modulo \(\beta\),
2. \(|\beta_i| > 1\) for every conjugate of \(\beta\).

If (1) and (2) hold, then:

1. \((\beta, A)\)-representations are unique.
2. There exists a simple algorithm for computing them.
3. To decide the GNS property, it suffices to check finitely many elements \(x\).

Situation in \(\mathbb{Z}\):

- No GNS with radix 2.
- Radix \(-2\): The only good alphabets are \(\pm\{0,1\}\).
- Good alphabets for 3: Difficult open question.
Kátai, Szabó, 1975: Classification of GNSs in $\mathbb{Z}[i]$ with $A = \{0, 1, \ldots, n\}$. (So-called canonical number systems (CNS).)

Steidl, 1989: Classification of all $\beta \in \mathbb{Z}[i]$ which admit at least one GNS $(\beta, A)$.

Kátai, 1992: Generalisation to $\mathcal{O}_K$ for imaginary quadratic $K$.

K., 2015: The analogue for Hurwitz and Lipschitz integral quaternions.

In the latter three cases, the statement is as follows:

A GNS with radix $\beta$ exists if and only if $|\beta| > 1$ and $|\beta - 1| \neq 1$. 
Definition (GNS on a lattice, Vince, 1993)

Let $\Lambda$ be a $\mathbb{Z}$-lattice and $L$ a linear operator on $\Lambda$. (For our purposes WLOG $\Lambda = \mathbb{Z}^d$ and $L \in \mathbb{Z}^{d \times d}$.) Let $A \ni 0$ be a finite subset of $\Lambda$. We call $(\beta, A)$ a GNS on $\Lambda$ if every nonzero $x \in \Lambda$ has a unique representation of the form

$$x = \sum_{i=0}^{N} L^i a_i,$$

where $N \in \mathbb{N}_0$, $a_i \in A$, $a_N \neq 0$.

Again: If $(L, A)$ is a GNS, then:

1. $A$ is a FRS modulo $L$;
   - in particular, $|A| = |\det L|$.
2. $L$ is expansive, i.e. $\rho(L^{-1}) < 1$;
3. $\det(L - I) \neq \pm 1$. “Unit condition.”
If $L$ is a radix of a GNS, then $L$ is expansive ($\rho(L^{-1}) < 1$).

For expansive $L$, there is a vector norm $\| \cdot \|$ such that $\|L^{-1}\| < 1$; this again gives an algorithm for checking the GNS property.

Germán, Kovács, 2007: If $\rho(L^{-1}) < 1/2$, then $L$ is a radix of some GNS.

K., 2018: If $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue, then $L$ is a radix of some GNS.
**Question:** Given a radix $L$, how many $\mathcal{A}$ are there such that $(L, \mathcal{A})$ is a GNS?

Matula, 1978: In $\mathbb{Z}$: If $|\beta| > 2$, then $\beta$ is a radix of infinitely many GNSs. (For $-2$ there are two GNSs, otherwise zero.)

**Question 2:** Can the suitable alphabets be “arbitrarily sparse” in the sense that all nonzero digits are far from the origin?
Theorem (Kovács, K.)

If \( L \in \mathbb{Z}^{d \times d} \) satisfies \( \rho(L^{-1}) < 1/2 \), then there are infinitely many \( A \) such that \((L, A)\) is a GNS in \( \mathbb{Z}^d \).

If one excludes \( L = (-2) \) in one dimension, it suffices to assume \( \rho(L^{-1}) \leq 1/2 \) with 2 not an eigenvalue.

In a number field order \( \mathcal{O} \): \( \beta \) is a radix of infinitely many GNSs iff it is a radix of at least one GNS and \( |N(\beta)| \geq 3 \).

**Conjecture:** If \( |\det L| \neq 2 \), then there are infinitely many GNSs for \( L \) if and only if there is at least one.
Theorem (K.)

Suppose that \( \rho(L^{-1}) < 1/2 \) and 2 is not an eigenvalue of \( L \). Then there exists a family of arbitrarily sparse GNSs except for the case when every eigenvalue of \( L \) is either an integer or a non-real algebraic number of degree 2, and has geometric multiplicity 1.

The proof is based on a clever choice of infinitely many different but related vector norms.
Theorem (Kovács, K.)

Let $L \in \mathbb{Z}^{2 \times 2}$ with non-real eigenvalues be given. Consider the family of all digit sets $A \subset \mathbb{Z}^2$ such that $(L, A)$ is a GNS.

1. The family is empty if and only if $\det L = 1$ or $\det(L - I) = \pm 1$.
2. The family is nonempty but finite if and only if $\det L = 2$ and $\det(L - I) \neq \pm 1$.
3. In all other cases, the family is infinite, i.e. there are infinitely many digit sets $A$ such that $(L, A)$ is a GNS.

This was the hardest part – we needed to develop a new general strategy to handle this case of dimension two.
Thank you for your attention (and for all your eventual questions)!
I also happily answer questions sent to krasensky (at) seznam.cz.