Lattice-based number systems with the same radix

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Contents

Definitions

Let R be a ring, $\beta \in R$ (*radix*) and $\mathcal{A} \ni 0$ a finite subset of R (*alphabet*). A $(\beta, \mathcal{A})$-*representation* of $x \in R$ is

$$x = \sum_{i=0}^{N} \beta^i a_i, \qquad \text{where } N \in \mathbb{N}_0, a_i \in \mathcal{A}, a_N \neq 0.$$

Definition

We call $(\beta, \mathcal{A})$ a *number system (GNS)* on R if every nonzero $x \in R$ has a **unique** $(\beta, \mathcal{A})$-representation.

Which of the following are GNSs (in $\mathbb{Z}$ or $\mathbb{Z}[\mathrm{i}]$)?
$(10, \{0, \dots, 9\})$; $(2, \{0, 1\})$; $(2, \{-1, 0, 1\})$; $(-2, \{0, 1\})$; $(3, \{-1, 0, 1\})$; $(10, \{-5, \dots, 5\})$; $(1 + \mathrm{i}, \{0, 1\})$; $(-1 + \mathrm{i}, \{0, 1\})$.

Answer: The negabinary, the balanced ternary and the Penney number system.

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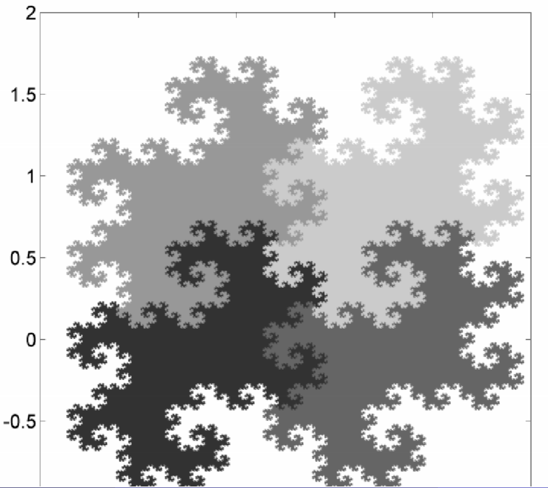
Answer: The negabinary, the balanced ternary and the Penney number system.

Penney, 1965: $(-1+\mathrm{i}, \{0,1\})$ is a GNS on $\mathbb{Z}[\mathrm{i}]$.

(For example, $-1 = \beta^4 + \beta^3 + \beta^2 + 1$ for $\beta = -1 + \mathrm{i}$.)

Negative powers $\sum_{j=-\infty}^{N} \beta^j a_j$ allow to represent all of $\mathbb{C}$:



Let $(\beta, \mathcal{A})$ be a GNS in a number field order ($\mathcal{O} \subset \mathcal{O}_K$ for a number field K). Then:

1. $\mathcal{A}$ is a FRS modulo β,
2. $|\beta_i| > 1$ for every conjugate of β.

If (1) and (2) hold, then:

1. $(\beta, \mathcal{A})$-representations are unique.
2. There exists a simple algorithm for computing them.
3. To decide the GNS property, it suffices to check finitely many elements x.

Situation in $\mathbb{Z}$:

- No GNS with radix 2.
- Radix -2: The only good alphabets are $\pm\{0, 1\}$.
- Good alphabets for 3: Difficult open question.

- Kátai, Szabó, 1975: Classification of GNSs in $\mathbb{Z}[\mathrm{i}]$ with $\mathcal{A} = \{0, 1, \ldots, n\}$. (So-called *canonical number systems (CNS)*.)

- Steidl, 1989: Classification of all $\beta \in \mathbb{Z}[\mathrm{i}]$ which admit at least one GNS $(\beta, \mathcal{A})$.
- Kátai, 1992: Generalisation to $\mathcal{O}_K$ for imaginary quadratic K.
- K., 2015: The analogue for Hurwitz and Lipschitz integral quaternions.

In the latter three cases, the statement is as follows:
A GNS with radix β exists if and only if $|\beta| > 1$ and $|\beta - 1| \neq 1$.

Definition (GNS on a lattice, Vince, 1993)

Let Λ be a $\mathbb{Z}$-lattice and L a linear operator on Λ. (For our purposes WLOG $\Lambda = \mathbb{Z}^d$ and $L \in \mathbb{Z}^{d\times d}$.) Let $\mathcal{A} \ni 0$ be a finite subset of Λ. We call $(\beta, \mathcal{A})$ a *GNS* on Λ if every nonzero $x \in \Lambda$ has a **unique** representation of the form

$$x = \sum_{i=0}^{N} L^i a_i, \qquad \text{where } N \in \mathbb{N}_0, a_i \in \mathcal{A}, a_N \neq 0.$$

Again: If $(L, \mathcal{A})$ is a GNS, then:

1. $\mathcal{A}$ is a FRS modulo L;
   - in particular, $|\mathcal{A}| = |\det L|$.
2. L is expansive, i.e. $\rho(L^{-1}) < 1$;
3. $\det(L - I) \neq \pm 1$. “Unit condition.”

- If L is a radix of a GNS, then L is *expansive* ($\rho(L^{-1}) < 1$).
- For expansive L, there is a vector norm $\|\cdot\|$ such that $\|L^{-1}\| < 1$; this again gives an algorithm for checking the GNS property.
- Germán, Kovács, 2007: If $\rho(L^{-1}) < 1/2$, then L is a radix of some GNS.
- K., 2018: If $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue, then L is a radix of some GNS.

Question: Given a radix L, how many $\mathcal{A}$ are there such that $(L, \mathcal{A})$ is a GNS?

Matula, 1978: In $\mathbb{Z}$: If $|\beta| > 2$, then β is a radix of infinitely many GNSs. (For -2 there are two GNSs, otherwise zero.)

Question 2: Can the suitable alphabets be "arbitrarily sparse" in the sense that all nonzero digits are far from the origin?

Theorem (Kovács, K.)

If $L \in \mathbb{Z}^{d\times d}$ satisfies $\rho(L^{-1}) < 1/2$, then there are infinitely many $\mathcal{A}$ such that $(L, \mathcal{A})$ is a GNS in $\mathbb{Z}^d$.

If one excludes $L = (-2)$ in one dimension, it suffices to assume $\rho(L^{-1}) \leq 1/2$ with 2 not an eigenvalue.

In a number field order $\mathcal{O}$: β is a radix of infinitely many GNSs iff it is a radix of at least one GNS and $|\mathrm{N}(\beta)| \geq 3$.

Conjecture: If $|\det L| \neq 2$, then there are infinitely many GNSs for L if and only if there is at least one.

Theorem (K.)

Suppose that $\rho(L^{-1}) < 1/2$ and 2 is not an eigenvalue of L. Then there exists a family of arbitrarily sparse GNSs except for the case when every eigenvalue of L is either an integer or a non-real algebraic number of degree 2, and has geometric multiplicity 1.

The proof is based on a clever choice of infinitely many different but related vector norms.

Theorem (Kovács, K.)

Let $L \in \mathbb{Z}^{2\times 2}$ with non-real eigenvalues be given. Consider the family of all digit sets $\mathcal{A} \subset \mathbb{Z}^2$ such that $(L, \mathcal{A})$ is a GNS.

1. *The family is empty if and only if* $\det L = 1$ *or* $\det(L - I) = \pm 1$.
2. *The family is nonempty but finite if and only if* $\det L = 2$ *and* $\det(L - I) \neq \pm 1$.
3. *In all other cases, the family is infinite, i.e. there are infinitely many digit sets $\mathcal{A}$ such that $(L, \mathcal{A})$ is a GNS.*

This was the hardest part – we needed to develop a new general strategy to handle this case of dimension two.

Thank you for your attention (and for all your eventual questions)!
I also happily answer questions sent to krasensky (at) seznam.cz.