On asymptotically automatic sequences

Jakub Konieczny

Camille Jordan Institute
Claude Bernard University Lyon 1

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The Thue–Morse(–Prouhet) sequence

The Thue–Morse sequence (discovered by Prouhet) $t\colon \mathbb{N} \to \{0, 1\}$,

$$0110100110010110100101100110 1001\ldots$$

is a (*the*?) paradigmatic example of an automatic sequence. It can be described in several equivalent ways:

1. Explicit formula: $t(n) = \begin{cases} 0 & \text{if } n \text{ is } \textit{evil} \text{ (i.e., sum of binary digits is even)}, \\ 1 & \text{if } n \text{ is } \textit{odious} \text{ (i.e., sum of binary digits is odd)}. \end{cases}$
2. Finite automaton:

3. Recurrence: $t(0) = 0, \quad t(2n) = t(n), \quad t(2n+1) = 1 - t(n)$.
4. Fixed point of a substitution: $0 \mapsto 01, \quad 1 \mapsto 10$.
5. Algebraic formal power series: If $T(z) = \sum_{n=0}^{\infty} t(n) z^n \in \mathbb{F}_2[[z]]$ then

$$z + (1+z)^2 T(z) + (1+z)^3 T(z)^2 = 0.$$

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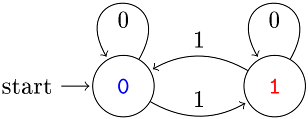

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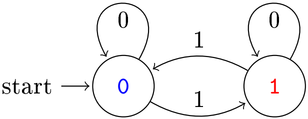

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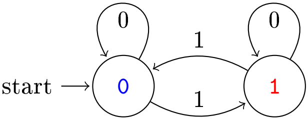

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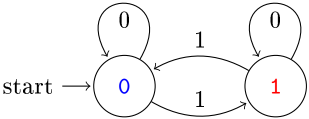

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Automatic sequences via finite automata

Some notation: We let k denote the base in which we work. $\longrightarrow$ e.g. $k = 10, k = 2$

- $\Sigma_k = \{0, 1, \ldots, k-1\}$, the set of digits in base k;
- Σ_k^* is the set of words over Σ_k, monoid with concatenation;
- for $n \in \mathbb{N}$, $(n)_k \in \Sigma_k^*$ is the base-k expansion of n; $\longrightarrow$ no leading zeros
- for $w \in \Sigma_k^*$, $[w]_k \in \mathbb{N}$ is the integer encoded by w.

A finite k-automaton consists of:

- a finite set of states S with a distinguished initial state s_0;
- a transition function $\delta \colon S \times \Sigma_k \to S$;
- an output function $\tau \colon S \to \Omega$.

Computing the sequence:

- Extend δ to a map $S \times \Sigma_k^*$ with $\delta(s, uv) = \delta(\delta(s, u), v)$ or $\delta(\delta(s, v), u)$;
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- The automaton above computes the Rudin–Shapiro sequence $(-1)^{\#\text{ of } 11 \text{ in } (n)_2}$.

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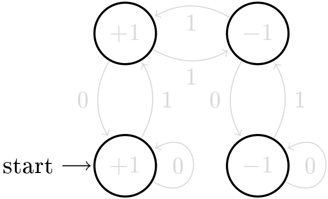

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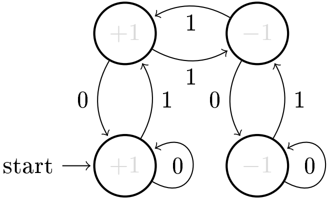

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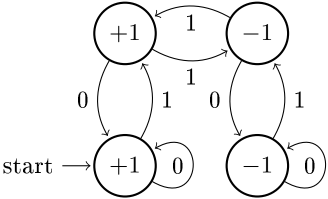

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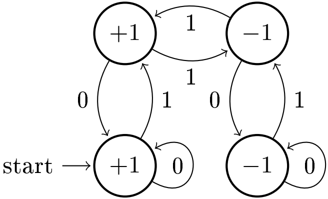

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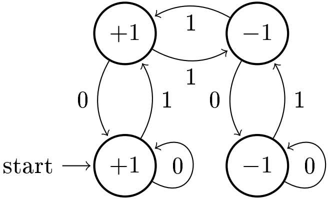

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Automatic sequences via kernels

Definition (Kernel)

Let $k \geq 2$ and let $f\colon \mathbb{N} \to \Omega$ be a sequence. Then the *k-kernel* of f is the set

$$\mathcal{N}_k(f) := \{f_{\alpha,m} \;:\; \alpha, m \in \mathbb{N},\; m < k^\alpha\}, \text{ where } f_{\alpha,m}(n) := f(k^\alpha n + m).$$

Examples:

- Let t be the Thue–Morse sequence, $t(n) = s_2(n) \bmod 2$. Then

$$\mathcal{N}_2(t) = \{t, 1 - t\}.$$

- Let $r(n)$ be the Rudin–Shapiro sequence, $r(n) = (-1)^{\#\text{ of } 11 \text{ in } (n)_2}$. Then $r(2n) = r(n)$, $r(4n+1) = r(n)$, $r(4n+3) = -r(2n+1)$. Hence,

$$\mathcal{N}_2(r) = \{\pm r, \pm r'\}, \text{ where } r'(n) = r(2n+1).$$

Proposition

A sequence f is k-automatic if and only if it has finite k-kernel, $\#\mathcal{N}_k(f) < \infty$.

Idea: Let $\mathcal{A} = (S, \delta, \Omega, \tau)$ be a (reduced) k-automaton computing f, reading least significant digits first. There is a bijection $S \longleftrightarrow \mathcal{N}_k(f)$.

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Asymptotics

- Two sequences $f, g\colon \mathbb{N} \to \Omega$ are *asymptotically equal*, denoted by
$$f(n) \simeq g(n),$$
if they differ on a set with asymptotic density zero:
$$\#\{n < N \,:\, f(n) \neq g(n)\}/N \to 0 \text{ as } N \to \infty.$$
- A sequence $f\colon \mathbb{N} \to \Omega$ is *asymptotically invariant under shift by* $m \in \mathbb{N}$ (or *asymptotically shift-invariant*, if m does not matter) if
$$f(n+m) \simeq f(n).$$
- A sequence $f\colon \mathbb{N} \to \Omega$ is *asymptotically periodic* if there is a periodic sequence $\tilde{f}\colon \mathbb{N} \to \Omega$ such that
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- Each asymptotically periodic sequence is asymptotically shift invariant.
- An asymptotically shift-invariant sequence is not necessarily asymptotically periodic, e.g. $f(n) = \lfloor \sqrt{n} \rfloor \bmod 2$.

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- Each asymptotically periodic sequence is asymptotically shift invariant.
- An asymptotically shift-invariant sequence is not necessarily asymptotically periodic, e.g. $f(n) = \lfloor \sqrt{n} \rfloor \bmod 2$.

Asymptotics

- Two sequences $f, g\colon \mathbb{N} \to \Omega$ are *asymptotically equal*, denoted by

$$f(n) \simeq g(n),$$

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Asymptotically automatic sequences

Definition

Let $k \geq 2$ be a base and let $f\colon \mathbb{N} \to \Omega$ be a sequence. Then f is *asymptotically k-automatic* if and only if $\mathcal{N}_k(f)/\simeq$ is finite. In other words, f is asymptotically k-automatic if there exist sequences $f_0, f_1, \ldots, f_{d-1}\colon \mathbb{N} \to \Omega$ such that for each $f' \in \mathcal{N}_k(f)$ there exists $0 \leq i < d$ such that $f'(n) \simeq f_i(n)$.

Example

Let $f\colon \mathbb{N} \to \Omega$ be k-automatic and let $g\colon \mathbb{N} \to \Omega$ be a sequence with $f(n) \simeq g(n)$. Then g is asymptotically k-automatic.

Example

Let $\lambda(n)$ denote the number of leading 1s in the binary expansion of n and

$$f(n) = f\Big(\big[\underbrace{11\ldots1}_{\lambda(n)}0**\cdots*\big]_2\Big) = \begin{cases} 1 & \text{if } \lambda(n) \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

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Why study the class of asymptotically automatic sequences?

- *"Because it's there."* — George Mallory
- Because it yields density versions of theorems on automatic sequences. (e.g. density version of Cobham's theorem)
- Because it sometimes comes up in applications. (e.g. upcoming work with O. Klurman on classification of automatic semigroups)
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Basic properties

Lemma (Closure under Cartesian products)

Let $f\colon \mathbb{N} \to \Omega$, $f'\colon \mathbb{N} \to \Omega'$ be asymptotically k-automatic. Then $f \times f'\colon \mathbb{N} \to \Omega \times \Omega'$ is also asymptotically k-automatic.

Lemma (Closure under coding)

Let $f\colon \mathbb{N} \to \Omega$ be asymptotically k-automatic and let $\rho\colon \Omega \to \Omega'$ be any map. Then $\rho \circ f\colon \mathbb{N} \to \Omega'$ is also asymptotically k-automatic.

Corollary: Complex-valued asymptotically k-automatic sequences constitute a ring.

Lemma (Passing to arithmetic progressions)

Let $f\colon \mathbb{N} \to \Omega$ be a sequence.

- *If f is asymptotically k-automatic then each restriction $f'(n) = f(an + b)$ $(a, b \in \mathbb{N})$ of f to an arithmetic progression is asymptotically k-automatic.*
- *Conversely, if there exists $a > 0$ such that $f'(n) = (an + b)$ is asymptotically k-automatic for each $0 \leq b < a$, then f is asymptotically k-automatic.*

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Automata

Recall that $\Sigma_k = \{0, 1, \ldots, k-1\}$ and Σ_k^* = words over Σ_k.

Definition

The k-kernel of a map $\phi \colon \Sigma_k^* \to \Omega$ is the set of maps $\Sigma_k^* \to \Omega$ given by

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$$f\left(k^{\alpha} n + [u]_k\right) = f\left([(n)_k u]_k\right) \simeq f_{\phi(u)}(n). \qquad (*)$$

- If the second condition holds, then $\#(\mathcal{N}_k(f)/\simeq) \leq d$, so we are done.
- Let f be asymptotically k-automatic, and let f_i be representatives of $\mathcal{N}_k(f)/\simeq$.
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Bases

Two integers $k, \ell \geq 2$ are *multiplicatively dependent* if they are both powers of the same integer: $k = m^a$, $\ell = m^b$ $(m, a, b \in \mathbb{N})$.

Fact

If $k, \ell \geq 2$ are multiplicatively dependent, then k-automatic sequences are the same as ℓ-automatic sequences. The same holds for asymptotically automatic sequences.

Idea: For simplicity, say $\ell = k^c$ for $c \in \mathbb{N}$. Then Σ_k^* can (almost) be identified with Σ_ℓ^* by grouping blocks of c symbols.

A sequence $f\colon \mathbb{N} \to \Omega$ is *eventually periodic* if there exist n_0 and $m > 0$, such that $f(n+m) = f(n)$ for all $n \geq n_0$.

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Let $f\colon \mathbb{N} \to \Omega$ be sequence that is eventually periodic. Then f is k-automatic for all bases $k \geq 2$.

Basic question: Given an automatic sequence f, in which bases is it automatic?

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Cobham's theorem

Theorem (Cobham, 1969)

Let $k, \ell \geq 2$ be two bases and let $f : \mathbb{N} \to \Omega$ be a sequence. If f is k-automatic and ℓ-automatic, then either

- *the bases k and ℓ are multiplicatively dependent, or*
- *the sequence f is eventually periodic.*

Corollary: The set of bases in which a given sequence is automatic is one of:

$$\emptyset, \qquad \{k^a : a \geq 1\} \text{ for some } k \geq 2, \qquad \mathbb{N}.$$

Intuition: A sequence cannot be automatic in two different bases (except for trivial cases).

Example

There is no 3-automaton which computes the Thue–Morse sequence.

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Let $k, \ell \geq 2$ be two bases and let $f\colon \mathbb{N} \to \Omega$ be a sequence. If f is k-automatic and ℓ-automatic, then either

- *the bases k and ℓ are multiplicatively dependent, or*
- *the sequence f is eventually periodic.*

Corollary: The set of bases in which a given sequence is automatic is one of:

$$\emptyset, \qquad \{k^a : a \geq 1\} \text{ for some } k \geq 2, \qquad \mathbb{N}.$$

Intuition: A sequence cannot be automatic in two different bases (except for trivial cases).

Example

There is no 3-automaton which computes the Thue–Morse sequence.

Generalisations of Cobham's theorem

Analogues of Cobham's theorem are known in the following contexts:

- Multidimensional sequences [Semenov 1977].
- Morphic sequences [Durand 2011].
- Fractals [Adamczewski, Bell 2011].
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- etc., etc., ...

Instead of *one* sequence $f\colon \mathbb{N} \to \Omega$, we can consider *two* sequences $f, g\colon \mathbb{N} \to \Omega$ that are k- and ℓ-automatic respectively, and which are "close enough". Cobham's theorem continues to hold *mutatis mutandis* when the assumption that $f = g$ is weakened to:

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Density version of Cobham's theorem

Theorem (Byszewski, K. 2017)

Let $k, \ell \geq 2$ be two multiplicatively independent bases. Let $f\colon \mathbb{N} \to \Omega$ be a k-automatic sequence and let $g\colon \mathbb{N} \to \Omega$ be an ℓ-automatic sequence such that $f(n) \simeq g(n)$. Then f and g are asymptotically periodic.

Example (Least significant digit of $n!$)

- Let $\ell_k(n)$ denote the first non-zero digit of n in base k, e.g. $\ell_{10}(10!) = \ell_{10}(3628800) = 8$.
- The sequences $\ell_k(n!)$ were studied by Deshouillers and Ruzsa, among others.
- Interesting feature: If k is a prime power then $\ell_k(n!)$ is k-automatic.
- More generally, let $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorisation of k, where

  $$\alpha_1(p_1 - 1) \geq \alpha_2(p_2 - 1) \geq \cdots \geq \alpha_r(p_r - 1).$$

  The sequence $\ell_k(n!)$ is p_1-automatic as long as $\alpha_1(p_1 - 1) \neq \alpha_2(p_2 - 1)$.
- Rationale: $\nu_p(n!) = \dfrac{n - s_p(n)}{p-1} \approx \dfrac{n}{p-1}$, so we expect that $\ell_k(n!) \equiv 0 \bmod k/p_1^{\alpha_1}$.
- For $k = 12$ we have $\alpha_1(p_1 - 1) = \alpha_2(p_2 - 1) = 2$. Deshouillers and Ruzsa showed that $\ell_{12}(n!) \simeq f(n)$ for a 3-automatic sequence $f\colon \mathbb{N} \to \{4, 8\}$. Also, $1_y(\ell_{12}(n!))$ is not automatic for $y = 3, 6, 9$, and in particular, $\ell_{12}(n!)$ is not automatic.
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- It follows from density Cobham's theorem that $1_y(\ell_{12}(n!))$ is not automatic for $y = 4, 8$.

Asymptotic versions of Cobham's theorem

Theorem (K. 2022)

Let $k, \ell \geq 2$ be two multiplicatively independent bases. Let $f\colon \mathbb{N} \to \Omega$ be a sequence that is asymptotically k-automatic and asymptotically ℓ-automatic.
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Let $k, \ell \geq 2$ be two multiplicatively independent bases. Let $f\colon \mathbb{N} \to \Omega$ be a sequence that is (classically) k-automatic and asymptotically ℓ-automatic.
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Asymptotic Cobham's theorem $\Longrightarrow$ Density Cobham's theorem.

Let $f\colon \mathbb{N} \to \Omega$ be a k-automatic sequence and let $g\colon \mathbb{N} \to \Omega$ be an ℓ-automatic sequence such that $f(n) \simeq g(n)$. Then f is asymptotically ℓ-automatic. Hence, by asymptotic Cobham's theorem, f is asymptotically periodic. □

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One might hope for a joint generalisation of the two theorems from the last slide:

Conjecture

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Unfortunately(?), this is false.

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Let us order all integers of the form $2^\alpha 3^\beta$ in increasing order

$$\mathcal{H} := \{H_0 < H_1 < H_2 < \cdots\} := \left\{2^\alpha 3^\beta \,:\, \alpha, \beta \geq 0\right\} = \{1, 2, 3, 4, 6, 8, 9, 12, \ldots\}.$$

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Fact: $H_{i+1}/H_i \to 1$ as $i \to \infty$. *Proof:* Kronecker equidistribution theorem.

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The sequence f is asymptotically 2- and 3-automatic.

In fact, $\#(\mathcal{N}_2(f)/\simeq) \leq 2$ and $\#(\mathcal{N}_3(f)/\simeq) \leq 2$.

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The sequence f is not asymptotically periodic.

- Suppose, for the sake of contradiction, that $f(n) \simeq \tilde{f}(n)$ for periodic $\tilde{f}$.
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- Since $f(2n) \simeq -f(n)$, also $\tilde{f}(2n) \simeq -\tilde{f}(n)$, so $c = -c$, but this is impossible.

Summary: A sequence that is asymptotically k- and ℓ-automatic for multiplicatively independent $k, \ell \geq 2$ does not need to be asymptotically periodic.

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$$f(n+1) \simeq f(n) \qquad f(2n) \simeq -f(n) \qquad f(3n) \simeq -f(n).$$

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Bases of automaticity

For a sequence $f\colon \mathbb{N} \to \Omega$, put $\mathcal{B}_{\mathrm{aut}}(f) := \{k \in \mathbb{N} : f \text{ is } k\text{-automatic}\}$.

Theorem (Cobham; alternative phrasing)

Let $f\colon \mathbb{N} \to \Omega$ be a sequence. Then $\mathcal{B}_{\mathrm{aut}}(f)$ one of:

- *the empty set $\emptyset$ (i.e., f is not automatic);*
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Let $f\colon \mathbb{N} \to \Omega$ be a sequence. There exists a vector space $V < \bigoplus_{p \in \mathcal{P}} \mathbb{Q}$ such that

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Are the following situations possible?

- $\mathcal{B}_{\text{asy}}(f) = \{2^a 3^b \ : \ a, b \in \mathbb{N}\}$ (we know: $\mathcal{B}_{\text{asy}}(f) \supseteq \{2^a 3^b \ : \ a, b \in \mathbb{N}\}$ is possible);
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Comments

- It is straightforward to generalise the example for bases 2 and 3 to any finite set of primes, *but* proving $f(pn) \simeq -f(n)$ requires a new argument.
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- To simplify: $\phi(0u) = \phi(u)$ for $u \in \Sigma_k^*$ and $\psi(0v) = \psi(u)$ for $v \in \Sigma_\ell^*$; thus

$$f\left(k^\alpha n + m\right) \simeq f_{\phi((m)_k)}(n) \qquad f\left(\ell^\beta n + m\right) \simeq f_{\psi((m)_\ell)}(n) \qquad \text{for each } m \in \mathbb{N}.$$

Proof of asymptotic Cobham's theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f : \mathbb{N} \to \Omega$ is asymptotically k-automatic and asymptotically ℓ-automatic;
- $f_0, f_1, \ldots, f_{d-1}$ are representatives of $\mathcal{N}_k(f)/\simeq$; $\vec{f} := (f_0, f_1, \ldots, f_{d-1}) \colon \mathbb{N} \to \Omega^d$;
- $g_0, g_1, \ldots, g_{e-1}$ are representatives of $\mathcal{N}_\ell(f)/\simeq$; $\vec{g} := (g_0, g_1, \ldots, g_{e-1}) \colon \mathbb{N} \to \Omega^e$;
- $\phi \colon \Sigma_k^* \to \Sigma_d$ is k-automatic and $f\left(k^\alpha n + [u]_k\right) \simeq f_{\phi(u)}(n)$;
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Proof of asymptotic Cobham's theorem

Lemma

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$, $\vec{y} \in \Omega^e$ *and* $E := \left\{ n \in \mathbb{N} \, : \, \vec{f}(\ell^\beta n) = \vec{x}, \; \vec{g}(k^\alpha n) = \vec{y} \right\}$. *Suppose that* $\bar{d}(E) > 0$. *Then* $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ *for all* $0 \leq m < \min(k^\alpha, \ell^\beta)$.

Proof of Lemma:

- $f(k^\alpha \ell^\beta n + m) = f_{\phi((m)_k)}(\ell^\beta n) = x_{\phi((m)_k)}$ for almost all $n \in E$.
- $f(k^\alpha \ell^\beta n + m) = g_{\psi((m)_\ell)}(k^\alpha n) = y_{\psi((m)_\ell)}$ for almost all $n \in E$.
- Since $\bar{d}(E) > 0$, there is at least *one* $n \in \mathbb{N}$ such that

$$x_{\phi((m)_k)} = f(k^\alpha \ell^\beta n + m) = y_{\psi((m)_\ell)}.$$

Proof of asymptotic Cobham's theorem

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Proof of asymptotic Cobham's theorem

Lemma

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$ *and* $\vec{y} \in \Omega^e$. *Suppose that*

$$\bar{d}\left(\left\{n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x},\ \vec{g}(k^\alpha n) = \vec{y}\right\}\right) > 0. \qquad (*)$$

Then $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ *for all* $0 \leq m < \min(k^\alpha, \ell^\beta)$.

Corollary

Let $\vec{x} \in \Omega^d$. *The sequence* $x_{\phi((m)_k)}$ *is eventually periodic, provided that* $(*)$ *holds for arbitrarily large* $\alpha, \beta \in \mathbb{N}$ *for some* $\vec{y} \in \Omega^e$. *Call such* $\vec{x}$ *"good".*

- Directly by definition, $x_{\phi((m)_k)}$ is k-automatic and $y_{\psi((m)_\ell)}$ is ℓ-automatic.
- By Lemma, $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ is k- and ℓ-automatic.
- By Cobham's theorem, $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ is eventually periodic.

Let q be the least common multiple of periods from Corollary above. For ease of notation assume $x_{\phi((m)_k)}$ is genuinely periodic.

Proof of asymptotic Cobham's theorem

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Let $\vec{x} \in \Omega^d$. The sequence $x_{\phi((m)_k)}$ is eventually periodic, provided that $(*)$ holds for arbitrarily large $\alpha, \beta \in \mathbb{N}$ for some $\vec{y} \in \Omega^e$. Call such $\vec{x}$ "good".

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Let q be the least common multiple of periods from Corollary above. For ease of notation assume $x_{\phi((m)_k)}$ is genuinely periodic.

Proof of asymptotic Cobham's theorem

Corollary

The sequence $x_{\phi((m)_k)}$ *has period* q *for each "good"* $\vec{x} \in \Omega^d$.

Lemma

Let $n \in \mathbb{N}$. *Then*

$$f(n+q) = f(n),$$

provided that there exists a decomposition $n = k^\alpha n' + m$ *where* $m < k^\alpha - q$ *and* $\vec{x} := \vec{f}(n')$ *is "good".*

Proof: $f(k^\alpha n' + m + q) = x_{\phi((m+q)_k)} = x_{\phi((m)_k)} = f(k^\alpha n' + m)$.

Lemma

For asymptotically almost all n, *there exists a decomposition* $n = k^\alpha n' + m$ *where* $n', m, \alpha \in \mathbb{N}$, $m < k^\alpha - q$, $\vec{f}(n')$ *is "good".*

Proof idea: For each $\alpha < \log_k n$, there is a positive chance to find the decomposition.

Corollary

The sequence $f(n)$ *is asymptotically invariant under shift by* q, *QED.*

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Proof of "mixed" Cobham's theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f\colon \mathbb{N} \to \Omega$ is k-automatic and asymptotically ℓ-automatic;
- By previous theorem, f is asymptotically invariant under shift by some $q \geq 1$;
- To simplify, assume that $q = 1$.

Lemma

Let $g\colon \mathbb{N} \to \{0,1\}$ be a k-automatic sequence with $g(n) \simeq 0$. Then there is $n_0 \in \mathbb{N}$ with

$$g(k^\alpha n_0 + m) = 0 \qquad \text{for all } \alpha \in \mathbb{N},\ 0 \leq m < k^\alpha.$$

Proof idea: Pick an automaton computing g, reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick $v \in \Sigma_k^*$ such that $\delta(s_0, v)$ lies in a strongly connected component, and put $n_0 = [v]_k$.

- Let n_0 be the constant from the Lemma applied to the k-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n+1) \neq f(n), \\ 0 & \text{if } f(n+1) = f(n). \end{cases}$$

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- Let n_0 be the constant from the Lemma applied to the k-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n+1) \neq f(n), \\ 0 & \text{if } f(n+1) = f(n). \end{cases}$$

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- Let n_0 be the constant from the Lemma applied to the k-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n+1) \neq f(n), \\ 0 & \text{if } f(n+1) = f(n). \end{cases}$$

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- Let n_0 be the constant from the Lemma applied to the k-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n+1) \neq f(n), \\ 0 & \text{if } f(n+1) = f(n). \end{cases}$$

Proof of "mixed" Cobham's theorem

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Proof of "mixed" Cobham's theorem

Reminder about assumptions and notation:

- $f\colon \mathbb{N} \to \Omega$ is k-automatic and asymptotically ℓ-automatic;
- f is constant on each interval $[k^\alpha n_0, k^\alpha(n_0+1))$.

Fact: The sequence $f(k^\alpha n_0)$ is eventually periodic with respect to α.

- To simplify: assume that $f(k^\alpha n_0) =: c$ is constant.
- Thus $f(n) = c$ for $n \in [k^\alpha n_0, k^\alpha(n_0+1))$ and $\alpha \in \mathbb{N}$.
- In other words, $f(n) = c$ for all n such that

$$\{\log_k(n)\} \in [\mu_0, \mu_0 + \delta),$$

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Definition

Let $f\colon \mathbb{N} \to \Omega$, $\omega \in \Omega$. The *(asymptotic / logarithmic) frequency* of ω if f is:

$$\operatorname{freq}(f;\omega) := \lim_{N\to\infty} \frac{1}{N} \cdot \#\left\{n < N \,:\, f(n) = \omega\right\},$$

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Proposition (Frequencies of symbols in automatic sequences)

Let $f\colon \mathbb{N} \to \Omega$ be automatic and $\omega \in \Omega$. Then

- the logarithmic frequency $\operatorname{freq}_{\log}(f;\omega)$ exists;
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$$(n)_2 = u_1^{(n)} u_2^{(n)} \cdots u_{r(n)}^{(n)} v^{(n)},$$

where $r(n) \in \mathbb{N}$, each $u_i^{(n)}$ ends with 1, $|u_i^{(n)}|_1 = i$, and $|v^{(n)}|_1 \leq r(n)$.

- We always have $r(2n) = r(n)$, and the expansion of $2n$ takes the form

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Classification problems

General questions: Fix the base $k \geq 2$.

- Given a sequence $f\colon \mathbb{N} \to \Omega$, decide if it is k-automatic.
- Given a class of sequences $\mathcal{F}$, find all $f \in \mathcal{F}$ which are k-automatic.

Definition

A set $E \subset \mathbb{N}$ is k-automatic if 1_E is k-automatic.

- Given a class $\mathcal{S}$ of subsets of $\mathbb{N}$, find all $E \in \mathcal{S}$ that are k-automatic.

Examples:

- *Cobham's theorem:* If $k, \ell \in \mathbb{N}$ are multiplicatively independent, then an ℓ-automatic sequence is k-automatic if and only if it is eventually periodic.
- *Primes and squares:* It is a standard exercise that the set of the primes and the set of the squares are not automatic. In fact, the set $\{p(n) : n \in \mathbb{N}\}$ of values of a polynomial p is automatic if and only if $\deg p = 1$.
- *Generalised polynomials:* Allouche and Shallit showed that sequences of the form $(\lfloor \alpha n + \beta \rfloor \bmod q)_{n=0}^{\infty}$ are automatic if and only if they are periodic.
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Classification of automatic multiplicative sequences

General fact

Fix a prime p. Each non-zero multiplicative sequence f has a unique representation

$$f(n) = h(\nu_p(n)) \cdot g(n/p^{\nu_p(n)}), \qquad (\dagger)$$

where $h(0) = 1$ and $g(pn) = 0$ for all n. Additionally, g is multiplicative.

Theorem (K., Lemańczyk, Müllner 2020)

Fix $k \geq 2$ and let $f : \mathbb{N} \to \mathbb{C}$ be a non-zero multiplicative sequence.

- *If k is a power of a prime p then f is k-automatic iff h and g given by $(\dagger)$ are eventually periodic. (In this case, g must be either periodic or eventually zero.)*
- *If k has ≥ 2 prime divisors then f is k-automatic iff f is eventually periodic.*

Remark: Conversely, each sequence f of the form described above is both k-automatic and multiplicative.

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Classification of automatic multiplicative sequences

Theorem (K.)

Fix $k \geq 2$ and let $f \colon \mathbb{N} \to \mathbb{C}$ be an asymptotically automatic multiplicative sequence. Then there exists $\chi \colon \mathbb{N} \to \mathbb{C}$ that is either a Dirichlet character or identically 0, such that $f(p^\alpha) = \chi(p^a)$ for all sufficiently large primes p and all $\alpha \in \mathbb{N}$.

Proof ideas:

- Key ingredient [Klurman 2017]: If f is finitely-valued, multiplicative and asymptotically invariant under a shift then $f \simeq 0$ or f is periodic.
- We can use old tricks to assume that f is *completely* multiplicative.
- Like earlier, we can find $f_0, f_1, \dots, f_{d-1}$ and k-automatic $\phi \colon \Sigma_k^* \to \Sigma_d$ such that $f(k^{|u|}n + [u]_k) \simeq f_{\phi(u)}(n)$ for $u \in \Sigma_k^*$. To simplify, assume that $\phi(0u) = \phi(u)$.
- If $f(q) \neq 0$ and $\phi(qm) = \phi(qm')$ then $\phi(m) = \phi(m')$:

$$f_{\phi(m)}(n) \simeq f(k^i n + m) = f(q)^{-1} f(k^i qn + qm) \simeq f(q)^{-1} f_{\phi(qm)}(qn)$$
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- For each $q \in \mathbb{N}$ without small prime factors, there exists $\hat{q} \in \mathbb{N}$ such that $\phi(\hat{q}q) = \phi(1)$.
- The last two items imply that if $\phi(q) = \phi(q')$ and $\phi(r) = \phi(r')$ then $\phi(qr) = \phi(q'r')$.
- Define a semigroup operation $\odot$ on (a subset of) Σ_d by $\phi(q) \odot \phi(r) = \phi(qr)$.
- Apply classification of automatic multiplicative sequences to conclude that ϕ is periodic (on integers without small prime factors).
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- Combining the last item with the fact from [Klurman 2017] finishes the argument.

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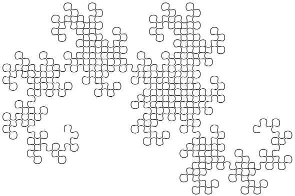

Automatic semigroups

General fact

Let p be a prime and let E be a p-automatic set. Then E can be decomposed as

$$E = E_0 \cup pE_1 \cup p^2 E_2 \cup \dots, \qquad (\dagger)$$

the sequence $E_0, E_1, E_2, \dots$ is eventually periodic, and $p \nmid n$ for all $n \in E_i$.

Theorem (Klurman, K. 2023+)

Let $k \geq 2$ and let $E \subset \mathbb{N}$ be a k-automatic semigroup. Assume further that E contains an infinite pairwise coprime subset.

- *If k is a power of a prime p then for each $i \geq 0$, the sets E_i are asymptotically periodic.*
- *If k has ≥ 2 prime divisors then E is asymptotically periodic.*

Recall: When all elements of E are allowed to share a factor, we get examples of the type $E = mX \cup m^2\mathbb{N}$, so the assumption cannot be removed. Not all sets of the above form are semigroups, but specifying which are is more mundane than difficult.

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Automatic multiplicatively stable sets

Definition

- For $E \subset \mathbb{N}$ and $m \in \mathbb{N}$ we let $E/m = \{n \in \mathbb{N} : mn \in E\}$. Note: $(mE)/m = E$.

Proposition

Let $E \subset \mathbb{N}$ be a k-automatic set. There exists a constant $\Delta \in \mathbb{N}$ with the property that for each $q \in \mathbb{N}$ with $\gcd(q, k) = 1$ and each $c \in \mathbb{Z}$, if we put $q' := \gcd(q, \Delta)$ then

$$d_{\log}((E - c)/q) = d_{\log}((E - c)/q').$$

Observation

Let $E \subset \mathbb{N}$ be a k-automatic semigroup, $q \in E$ and $\gcd(q, k\Delta) = 1$. Then $E/q \simeq E$.

Proof:

- Since E is a semigroup and $q \in E$, we have $E/q \supseteq E$.
- Since $\gcd(q, \Delta) = 1$, we have $d_{\log}(E/q) = d_{\log}(E)$.
- Combining the two points above: $d_{\log}(E/q \triangle E) = d_{\log}(E/q) - d_{\log}(E) = 0$. □

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- For $E \subset \mathbb{N}$ and $m \in \mathbb{N}$ we let $E/m = \{n \in \mathbb{N} : mn \in E\}$. Note: $(mE)/m = E$.

Proposition

Let $E \subset \mathbb{N}$ be a k-automatic set. There exists a constant $\Delta \in \mathbb{N}$ with the property that for each $q \in \mathbb{N}$ with $\gcd(q, k) = 1$ and each $c \in \mathbb{Z}$, if we put $q' := \gcd(q, \Delta)$ then

$$d_{\log}((E - c)/q) = d_{\log}((E - c)/q').$$

Observation

Let $E \subset \mathbb{N}$ be a k-automatic semigroup, $q \in E$ and $\gcd(q, k\Delta) = 1$. Then $E/q \simeq E$.

Proof:

- Since E is a semigroup and $q \in E$, we have $E/q \supseteq E$.
- Since $\gcd(q, \Delta) = 1$, we have $d_{\log}(E/q) = d_{\log}(E)$.
- Combining the two points above: $d_{\log}(E/q \triangle E) = d_{\log}(E/q) - d_{\log}(E) = 0$. □

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Multiplicative invariance

Definition

Let $E \subset \mathbb{N}$ be a set. We define the asymptotically invariant and reversible sets:

$$\operatorname{Inv}(E) := \{q \in \mathbb{N} : E/q \simeq E\},$$
$$\operatorname{Rev}(E) := \{q \in \mathbb{N} : q\mathbb{N} \cap \operatorname{Inv}(E) \neq \emptyset\}.$$

Theorem (Klurman, K. 2023+)

Let $k \geq 2$, let $E, F \subset \mathbb{N}$ be k-automatic sets with $F \subset \operatorname{Inv}(E)$ and $d_{\log}(F) > 0$.

- *If k is a power of a prime p then $E = E_0 \cup pE_1 \cup p^2E_2 \cup \dots$, where E_i are asymptotically periodic.*
- *If k has ≥ 2 prime divisors then E is asymptotically periodic.*

Proof ideas (slightly oversimplified)

- The set $\operatorname{Rev}(E)$ is periodic.
- We can construct a finite group $G_E := \operatorname{Rev}(E)/\operatorname{Inv}(E)$.
- The quotient map $\pi_E : \mathbb{N} \to G_E \cup \{0\}$ is k-automatic.
- The map π_E is periodic (by classification of automatic multiplicative sequences).
- The set E is asymptotically periodic.

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Asymptotically automatic sequences

Question

- Can we characterise pairs of k-automatic sets $E, F \subset \mathbb{N}$ with $F \subset \mathrm{Inv}(E)$?
- Can we use assumptions like $E/q \simeq E$ or $E/q \supseteq E$ when q is *not* coprime to k?

Example

Let E be 10-automatic set with $2 \in \mathrm{Inv}(E)$. Then 1_E is asymptotically 5-automatic;

$$1_E(5^\alpha n + m) \simeq 1_E(10^\alpha n + 2^\alpha m) \in \mathcal{N}_{10}(1_E(n))$$

for each $\alpha, m \in \mathbb{N}$ with $m < 5^\alpha$, and hence $\#\left(\mathcal{N}_5(1_E)/\simeq\right) \leq \#\left(\mathcal{N}_{10}(1_E)\right)$.

Corollary

If $E \subset \mathbb{N}$ is a 10-automatic set with $2 \in \mathrm{Inv}(E)$ then E is asymptotically periodic.

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