On asymptotically automatic sequences

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The Thue–Morse sequence

The Thue–Morse sequence (discovered by Prouhet) \( t : \mathbb{N} \rightarrow \{0, 1\} \),

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is a (the?) paradigmatic example of an automatic sequence. It can be described in several equivalent ways:

1. Explicit formula: \( t(n) = \begin{cases} 0 & \text{if } n \text{ is evil (i.e., sum of binary digits is even)}, \\ 1 & \text{if } n \text{ is odious (i.e., sum of binary digits is odd)}. \end{cases} \)

2. Finite automaton:

![Finite automaton diagram]

3. Recurrence: \( t(0) = 0, \quad t(2n) = t(n), \quad t(2n + 1) = 1 - t(n). \)

4. Fixed point of a substitution: \( 0 \mapsto 01, \quad 1 \mapsto 10. \)

5. Algebraic formal power series: If \( T(z) = \sum_{n=0}^{\infty} t(n)z^n \in \mathbb{F}_2[[z]] \) then

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z + (1 + z)^2 T(z) + (1 + z)^3 T(z)^2 = 0.
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 \__\|___ \__\|___
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![Finite Automaton Diagram](image)

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Automatic sequences via finite automata

Some notation: We let $k$ denote the base in which we work.

- $\Sigma_k = \{0, 1, \ldots, k-1\}$, the set of digits in base $k$;
- $\Sigma_k^*$ is the set of words over $\Sigma_k$, monoid with concatenation;
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A finite $k$-automaton consists of:

- a finite set of states $S$ with a distinguished initial state $s_0$;
- a transition function $\delta : S \times \Sigma_k \to S$;
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Computing the sequence:

- Extend $\delta$ to a map $S \times \Sigma_k^*$ with $\delta(s, uv) = \delta(\delta(s, u), v)$ or $\delta(\delta(s, v), u)$;
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Intuition: Automatic $\iff$ Computable by a finite device.
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Automatic sequences via kernels

**Definition (Kernel)**

Let $k \geq 2$ and let $f : \mathbb{N} \to \Omega$ be a sequence. Then the $k$-kernel of $f$ is the set

$$\mathcal{N}_k(f) := \{f_{\alpha,m} : \alpha, m \in \mathbb{N}, m < k^{\alpha}\},$$

where $f_{\alpha,m}(n) := f(k^{\alpha}n + m)$.

**Examples:**

- Let $t$ be the Thue–Morse sequence, $t(n) = s_2(n) \mod 2$. Then
  $$\mathcal{N}_2(t) = \{t, 1 - t\}.$$

- Let $r(n)$ be the Rudin–Shapiro sequence, $r(n) = (-1)^\# \text{ of } 11 \text{ in } (n)_2$. Then
  $r(2n) = r(n)$, $r(4n + 1) = r(n)$, $r(4n + 3) = -r(2n + 1)$. Hence,
  $$\mathcal{N}_2(r) = \{\pm r, \pm r'\},$$
  where $r'(n) = r(2n + 1)$.

**Proposition**

A sequence $f$ is $k$-automatic if and only if it has finite $k$-kernel, $\#\mathcal{N}_k(f) < \infty$.

**Idea:** Let $A = (S, \delta, \Omega, \tau)$ be a (reduced) $k$-automaton computing $f$, reading least significant digits first. There is a bijection $S \leftrightarrow \mathcal{N}_k(f)$. 
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Asymptotics

- Two sequences \( f, g : \mathbb{N} \to \Omega \) are *asymptotically equal*, denoted by
  \[ f(n) \simeq g(n), \]
  if they differ on a set with asymptotic density zero:
  \[ \frac{\# \{ n < N : f(n) \neq g(n) \}}{N} \to 0 \text{ as } N \to \infty. \]

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**Example**

- Each asymptotically periodic sequence is asymptotically shift invariant.
- An asymptotically shift-invariant sequence is not necessarily asymptotically periodic, e.g. \( f(n) = \lfloor \sqrt{n} \rfloor \mod 2 \).
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Asymptotically automatic sequences

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Let $k \geq 2$ be a base and let $f : \mathbb{N} \rightarrow \Omega$ be a sequence. Then $f$ is asymptotically $k$-automatic if and only if $\mathcal{N}_k(f) / \simeq$ is finite. In other words, $f$ is asymptotically $k$-automatic if there exist sequences $f_0, f_1, \ldots, f_{d-1} : \mathbb{N} \rightarrow \Omega$ such that for each $f' \in \mathcal{N}_k(f)$ there exists $0 \leq i < d$ such that $f'(n) \simeq f_i(n)$.

**Example**
Let $f : \mathbb{N} \rightarrow \Omega$ be $k$-automatic and let $g : \mathbb{N} \rightarrow \Omega$ be a sequence with $f(n) \simeq g(n)$. Then $g$ is asymptotically $k$-automatic.

**Example**
Let $\lambda(n)$ denote the number of leading 1s in the binary expansion of $n$ and

$$f(n) = f\left([\underbrace{11\ldots10}_{\lambda(n)} * * \cdots *]_2\right) = \begin{cases} 1 & \text{if } \lambda(n) \text{ is prime}, \\ 0 & \text{otherwise}. \end{cases}$$

Then $f$ is asymptotically 2-automatic.
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Let $k \geq 2$ be a base and let $f : \mathbb{N} \to \Omega$ be a sequence. Then $f$ is *asymptotically $k$-automatic* if and only if $\mathcal{N}_k(f)/\simeq$ is finite. In other words, $f$ is asymptotically $k$-automatic if there exist sequences $f_0, f_1, \ldots, f_{d-1} : \mathbb{N} \to \Omega$ such that for each $f' \in \mathcal{N}_k(f)$ there exists $0 \leq i < d$ such that $f'(n) \simeq f_i(n)$.

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Motivation

Why study the class of asymptotically automatic sequences?

- “Because it’s there.” — George Mallory

- Because it yields density versions of theorems on automatic sequences.
  (e.g. density version of Cobham’s theorem)

- Because it sometimes comes up in applications.
  (e.g. upcoming work with O. Klurman on classification of automatic semigroups)

- To better understand relations between properties of automatic sequences.
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Basic properties

**Lemma (Closure under Cartesian products)**

Let \( f : \mathbb{N} \to \Omega, f' : \mathbb{N} \to \Omega' \) be asymptotically \( k \)-automatic. Then \( f \times f' : \mathbb{N} \to \Omega \times \Omega' \) is also asymptotically \( k \)-automatic.

**Lemma (Closure under coding)**

Let \( f : \mathbb{N} \to \Omega \) be asymptotically \( k \)-automatic and let \( \rho : \Omega \to \Omega' \) be any map. Then \( \rho \circ f : \mathbb{N} \to \Omega' \) is also asymptotically \( k \)-automatic.

**Corollary:** Complex-valued asymptotically \( k \)-automatic sequences constitute a ring.

**Lemma (Passing to arithmetic progressions)**

Let \( f : \mathbb{N} \to \Omega \) be a sequence.

- If \( f \) is asymptotically \( k \)-automatic then each restriction \( f''(n) = f(an + b) \) \((a, b \in \mathbb{N})\) of \( f \) to an arithmetic progression is asymptotically \( k \)-automatic.
- Conversely, if there exists \( a > 0 \) such that \( f''(n) = (an + b) \) is asymptotically \( k \)-automatic for each \( 0 \leq b < a \), then \( f \) is asymptotically \( k \)-automatic.
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Automata

Recall that $\Sigma_k = \{0, 1, \ldots, k - 1\}$ and $\Sigma_k^* = \text{words over } \Sigma_k$.

**Definition**

The $k$-kernel of a map $\phi: \Sigma_k^* \rightarrow \Omega$ is the set of maps $\Sigma_k^* \rightarrow \Omega$ given by

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where $\phi_v(u) := \phi(uv)$ for $u, v \in \Sigma_k^*$.

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**Lemma**

Fix a base $k \geq 2$. For a sequence $f: \mathbb{N} \rightarrow \Omega$, the following conditions are equivalent.

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$$f(k^\alpha n + [u]_k) = f([(n)_k u]_k) \simeq f_{\phi(u)}(n).$$

(*)

- If the second condition holds, then $\#(\mathcal{N}_k(f)/\simeq) \leq d$, so we are done.
- Let $f$ be asymptotically $k$-automatic, and let $f_i$ be representatives of $\mathcal{N}_k(f)/\simeq$.
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Two integers \( k, \ell \geq 2 \) are \textit{multiplicatively dependent} if they are both powers of the same integer: \( k = m^a, \ell = m^b \) \((m, a, b \in \mathbb{N})\).

\[ \text{Fact} \]

If \( k, \ell \geq 2 \) are multiplicatively dependent, then \( k \)-automatic sequences are the same as \( \ell \)-automatic sequences. The same holds for asymptotically automatic sequences.

\[ \text{Idea:} \] For simplicity, say \( \ell = k^c \) for \( c \in \mathbb{N} \). Then \( \Sigma_k^* \) can (almost) be identified with \( \Sigma_\ell^* \) by grouping blocks of \( c \) symbols.

A sequence \( f : \mathbb{N} \to \Omega \) is \textit{eventually periodic} if there exist \( n_0 \) and \( m > 0 \), such that \( f(n + m) = f(n) \) for all \( n \geq n_0 \).

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Cobham’s theorem

Theorem (Cobham, 1969)

Let $k, \ell \geq 2$ be two bases and let $f : \mathbb{N} \to \Omega$ be a sequence. If $f$ is $k$-automatic and $\ell$-automatic, then either

- the bases $k$ and $\ell$ are multiplicatively dependent, or
- the sequence $f$ is eventually periodic.

Corollary: The set of bases in which a given sequence is automatic is one of:

$$\emptyset, \quad \{k^a : a \geq 1\} \text{ for some } k \geq 2, \quad \mathbb{N}.$$

Intuition: A sequence cannot be automatic in two different bases (except for trivial cases).

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There is no 3-automaton which computes the Thue–Morse sequence.
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Generalisations of Cobham’s theorem

Analogues of Cobham’s theorem are known in the following contexts:

- Multidimensional sequences [Semenov 1977].
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- Let \( \ell_k(n) \) denote the first non-zero digit of \( n \) in base \( k \), e.g. \( \ell_{10}(10!) = \ell_{10}(3628800) = 8 \).
- The sequences \( \ell_k(n!) \) were studied by Deshouillers and Ruzsa, among others.
- Interesting feature: If \( k \) is a prime power then \( \ell_k(n!) \) is \( k \)-automatic.
- More generally, let \( k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the prime factorisation of \( k \), where
  \[
  \alpha_1(p_1 - 1) \geq \alpha_2(p_2 - 1) \geq \cdots \geq \alpha_r(p_r - 1).
  \]
  The sequence \( \ell_k(n!) \) is \( p_1 \)-automatic as long as \( \alpha_1(p_1 - 1) \neq \alpha_2(p_2 - 1) \).
- Rationale: \( \nu_p(n!) = \frac{n - s_p(n)}{p - 1} \approx \frac{n}{p - 1} \), so we expect that \( \ell_k(n!) \equiv 0 \mod k/p_1^{\alpha_1} \).
- For \( k = 12 \) we have \( \alpha_1(p_1 - 1) = \alpha_2(p_2 - 1) = 2 \). Deshouillers and Ruzsa showed that \( \ell_{12}(n!) \simeq f(n) \) for a \( 3 \)-automatic sequence \( f : \mathbb{N} \to \{4, 8\} \). Also, \( 1_y(\ell_{12}(n!)) \) is not automatic for \( y = 3, 6, 9 \), and in particular, \( \ell_{12}(n!) \) is not automatic.
- It follows from density Cobham’s theorem that \( 1_y(\ell_{12}(n!)) \) is not automatic for \( y = 4, 8 \).
Asymptotic versions of Cobham’s theorem

**Theorem (K. 2022)**

Let $k, \ell \geq 2$ be two multiplicatively independent bases. Let $f : \mathbb{N} \to \Omega$ be a sequence that is asymptotically $k$-automatic and asymptotically $\ell$-automatic. Then $f$ is asymptotically shift invariant.

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Asymptotic Cobham’s theorem $\iff$ Density Cobham’s theorem.

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One might hope for a joint generalisation of the two theorems from the last slide:

**Conjecture**

Let $k, \ell \geq 2$ be two multiplicatively independent bases. Let $f : \mathbb{N} \to \Omega$ be a sequence that is asymptotically $k$-automatic and asymptotically $\ell$-automatic. Then $f$ is asymptotically periodic.

Unfortunately(?), this is false.

**Example**

Let us order all integers of the form $2^\alpha 3^\beta$ in increasing order

$$\mathcal{H} := \{H_0 < H_1 < H_2 < \cdots \} := \left\{2^\alpha 3^\beta : \alpha, \beta \geq 0 \right\} = \{1, 2, 3, 4, 6, 8, 9, 12, \ldots \}.$$

Let $H_i = 2^{\alpha_i} 3^{\beta_i}$ and define $f : \mathbb{N} \to \{-1, +1\}$ by

$$f(n) := (-1)^{\alpha_i + \beta_i} \text{ for } n \in [H_i, H_{i+1}) \text{ and } i \geq 0.$$ 

We will show that $f$ is asymptotically 2- and 3-automatic, but not asymptotically periodic.
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\[ f(n + 1) \simeq f(n), \quad f(2n) \simeq -f(n), \quad f(3n) \simeq -f(n). \]

- We only discuss \( f(2n) \simeq -f(n). \) Consider any \( n \in [H_i, H_{i+1}) \) with \( f(2n) = f(n). \)
- We have \( 2n \in [2H_i, 2H_{i+1}), \) where \( 2H_i =: H_j \in \mathcal{H} \text{ and } 2H_{i+1} =: H_j' \in \mathcal{H}. \)
- If \( 2n \in [H_j, H_{j+1}) \) then \( f(2n) = (-1)^{\alpha_i + \beta_i} = -f(n), \) so \( j' \geq j + 2. \)
- Since \( H_i < H_{j+1}/2 < H_{i+1} \) we have \( 2 \nmid H_{j+1}. \) Thus, \( H_{j+1} \) is a power of 3.
- Since \([H_j, H_{j'})\) cannot contain two powers of 3, we have \( j' = j + 2. \)
- Summarising, we have \( 2n \in \left[H_{j+1}, H_{j+2}\right) = \left[3^{\beta_{j+1}}, 3^{\beta_{j+1}}(1 + o(1))\right). \)
- Thus, the number of “bad” \( n\)'s in \( \left[\frac{1}{2} 3^\beta, \frac{1}{2} 3^{\beta+1}\right) \) is \( o(3^\beta). \) Take sum w.r.t. \( \beta. \) \( \square \)
Example in bases 2 and 3

Reminder:

\[ f(n + 1) \simeq f(n) \quad f(2n) \simeq -f(n) \quad f(3n) \simeq -f(n). \]

Corollary

The sequence \( f \) is asymptotically 2- and 3-automatic.

In fact, \( \# (N_2(f)/\simeq) \leq 2 \) and \( \# (N_3(f)/\simeq) \leq 2 \).

Lemma

The sequence \( f \) is not asymptotically periodic.

- Suppose, for the sake of contradiction, that \( f(n) \simeq \tilde{f}(n) \) for periodic \( \tilde{f} \).
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Summary: A sequence that is asymptotically \( k \)- and \( \ell \)-automatic for multiplicatively independent \( k, \ell \geq 2 \) does not need to be asymptotically periodic.
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Bases of automaticity

For a sequence \( f : \mathbb{N} \rightarrow \Omega \), put \( B_{\text{aut}}(f) := \{ k \in \mathbb{N} : f \text{ is } k\text{-automatic} \} \).

**Theorem (Cobham; alternative phrasing)**

Let \( f : \mathbb{N} \rightarrow \Omega \) be a sequence. Then \( B_{\text{aut}}(f) \) one of:

- the empty set \( \emptyset \) (i.e., \( f \) is not automatic);
- a geometric progression \( \{ k^a : a \geq 1 \} \) for some \( k \geq 2 \);
- all integers \( \mathbb{N} \) (i.e., \( f \) is eventually periodic).

In the same spirit, put \( B_{\text{asy}}(f) := \{ k \in \mathbb{N} : f \text{ is asymptotically } k\text{-automatic} \} \).

**Theorem (asymptotic variant of Cobham; alternative phrasing)**

Let \( f : \mathbb{N} \rightarrow \Omega \) be a sequence. Then one of the following holds:

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Let \( f : \mathbb{N} \to \Omega \) be a sequence. Then the set \( \mathcal{B}_{\text{asy}}(f) \) of bases with respect to which \( f \) is asymptotically automatic has the following closure properties:

- if \( k, \ell \in \mathcal{B}_{\text{asy}}(f) \) then \( k\ell \in \mathcal{B}_{\text{asy}}(f) \);
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- if \( k \in \mathcal{B}_{\text{asy}}(f) \), \( a \in \mathbb{Q}_+ \) and \( k^a \in \mathbb{N} \) then \( k^a \in \mathcal{B}_{\text{asy}}(f) \).

Corollary

Let \( f : \mathbb{N} \to \Omega \) be a sequence. There exists a vector space \( V < \bigoplus_{p \in \mathcal{P}} \mathbb{Q} \) such that

\[
\mathcal{B}_{\text{asy}}(f) = \{ k \in \mathbb{N}_{\geq 2} : (\nu_p(f))_{p \in \mathcal{P}} \in V \}.
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Conjecture: The converse is also true.
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**Question**

Are the following situations possible?

- $\mathcal{B}_{\text{asy}}(f) = \{2^a 3^b : a, b \in \mathbb{N}\}$ (we know: $\mathcal{B}_{\text{asy}}(f) \supseteq \{2^a 3^b : a, b \in \mathbb{N}\}$ is possible);
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**Comments**

- It is straightforward to generalise the example for bases 2 and 3 to any finite set of primes, *but* proving $f(pn) \simeq -f(n)$ requires a new argument.
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Proof of asymptotic Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f : \mathbb{N} \to \Omega$ is asymptotically $k$-automatic and asymptotically $\ell$-automatic;
- $f_0, f_1, \ldots, f_{d-1}$ are representatives of $\mathcal{N}_k(f)/\sim; \quad \vec{f} := (f_0, f_1, \ldots, f_{d-1}) : \mathbb{N} \to \Omega^d$;
- $g_0, g_1, \ldots, g_{e-1}$ are representatives of $\mathcal{N}_\ell(f)/\sim; \quad \vec{g} := (g_0, g_1, \ldots, g_{e-1}) : \mathbb{N} \to \Omega^e$;
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- $f_0, f_1, \ldots, f_{d-1}$ are representatives of $\mathcal{N}_k(f) / \simeq$; $\vec{f} := (f_0, f_1, \ldots, f_{d-1}) : \mathbb{N} \to \Omega^d$;
- $g_0, g_1, \ldots, g_{e-1}$ are representatives of $\mathcal{N}_\ell(f) / \simeq$; $\vec{g} := (g_0, g_1, \ldots, g_{e-1}) : \mathbb{N} \to \Omega^e$;
- $\phi : \Sigma^*_k \to \Sigma_d$ is $k$-automatic and $f (k^\alpha n + [u]_k) \simeq f_{\phi(u)}(n)$;
- $\psi : \Sigma^*_\ell \to \Sigma_e$ is $\ell$-automatic and $f (\ell^\beta n + [v]_\ell) \simeq f_{\psi(v)}(n)$;
- To simplify: $\phi(0u) = \phi(u)$ for $u \in \Sigma^*_k$ and $\psi(0v) = \psi(u)$ for $v \in \Sigma^*_\ell$; thus

$$f (k^\alpha n + m) \simeq f_{\phi((m)_k)}(n) \quad f (\ell^\beta n + m) \simeq f_{\psi((m)_\ell)}(n) \quad \text{for each } m \in \mathbb{N}.$$
Proof of asymptotic Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
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- $\phi : \Sigma_k^* \to \Sigma_d$ is $k$-automatic and $f(k^\alpha n + [u]_k) \simeq f_{\phi(u)}(n)$;
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- To simplify: $\phi(0u) = \phi(u)$ for $u \in \Sigma_k^*$ and $\psi(0v) = \psi(u)$ for $v \in \Sigma_\ell^*$; thus
  
  $f(k^\alpha n + m) \simeq f_{\phi((m)_k)}(n) \quad f(\ell^\beta n + m) \simeq f_{\psi((m)_{\ell})}(n)$  
  for each $m \in \mathbb{N}$. 
Proof of asymptotic Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
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\[ f(k^\alpha n + m) \simeq f_{\phi((m)_k)}(n) \quad f(\ell^\beta n + m) \simeq f_{\psi((m)_\ell)}(n) \quad \text{for each } m \in \mathbb{N}. \]
Proof of asymptotic Cobham’s theorem

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- $k, \ell \geq 2$ are multiplicatively independent integers;
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- To simplify: $\phi(0u) = \phi(u)$ for $u \in \Sigma_k^*$ and $\psi(0v) = \psi(u)$ for $v \in \Sigma_\ell^*$; thus
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Proof of asymptotic Cobham’s theorem

Lemma

Let \( \alpha, \beta \in \mathbb{N} \), \( \vec{x} \in \Omega^d \), \( \vec{y} \in \Omega^e \) and \( E := \{ n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x}, \vec{g}(k^{\alpha} n) = \vec{y} \} \). Suppose that \( \bar{d}(E) > 0 \). Then \( x_{\phi((m)_k)} = y_{\psi((m)_\ell)} \) for all \( 0 \leq m < \min(k^{\alpha}, \ell^\beta) \).

Proof of Lemma:

- \( f(k^{\alpha} \ell^\beta n + m) = f_{\phi((m)_k)}(\ell^\beta n) = x_{\phi((m)_k)} \) for almost all \( n \in E \).

- \( f(k^{\alpha} \ell^\beta n + m) = g_{\psi((m)_\ell)}(k^{\alpha} n) = y_{\psi((m)_\ell)} \) for almost all \( n \in E \).

- Since \( \bar{d}(E) > 0 \), there is at least one \( n \in \mathbb{N} \) such that

\[
x_{\phi((m)_k)} = f(k^{\alpha} \ell^\beta n + m) = y_{\psi((m)_\ell)}.
\]
Proof of asymptotic Cobham’s theorem

**Lemma**

Let \( \alpha, \beta \in \mathbb{N} \), \( \vec{x} \in \Omega^d \), \( \vec{y} \in \Omega^e \) and \( E := \left\{ n \in \mathbb{N} : f(\ell^\beta n) = \vec{x}, g(k^\alpha n) = \vec{y} \right\} \). Suppose that \( \bar{d}(E) > 0 \). Then \( x_{\phi((m)_k)} = y_{\psi((m)_\ell)} \) for all \( 0 \leq m < \min(k^\alpha, \ell^\beta) \).

**Proof of Lemma:**

- \( f(k^\alpha \ell^\beta n + m) = f_{\phi((m)_k)}(\ell^\beta n) = x_{\phi((m)_k)} \) for almost all \( n \in E \).

- \( f(k^\alpha \ell^\beta n + m) = g_{\psi((m)_\ell)}(k^\alpha n) = y_{\psi((m)_\ell)} \) for almost all \( n \in E \).

- Since \( \bar{d}(E) > 0 \), there is at least one \( n \in \mathbb{N} \) such that

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x_{\phi((m)_k)} = f(k^\alpha \ell^\beta n + m) = y_{\psi((m)_\ell)}.
\]
Proof of asymptotic Cobham’s theorem

**Lemma**

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$, $\vec{y} \in \Omega^e$ and $E := \{ n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x}, \vec{g}(k^\alpha n) = \vec{y} \}$. Suppose that $\bar{d}(E) > 0$. Then $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ for all $0 \leq m < \min(k^\alpha, \ell^\beta)$.

**Proof of Lemma:**

- $f(k^\alpha \ell^\beta n + m) = f_{\phi((m)_k)}(\ell^\beta n) = x_{\phi((m)_k)}$ for almost all $n \in E$.

- $f(k^\alpha \ell^\beta n + m) = g_{\psi((m)_\ell)}(k^\alpha n) = y_{\psi((m)_\ell)}$ for almost all $n \in E$.

- Since $\bar{d}(E) > 0$, there is at least one $n \in \mathbb{N}$ such that $x_{\phi((m)_k)} = f(k^\alpha \ell^\beta n + m) = y_{\psi((m)_\ell)}$. 

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Proof of asymptotic Cobham’s theorem

**Lemma**

Let \( \alpha, \beta \in \mathbb{N}, \vec{x} \in \Omega^d, \vec{y} \in \Omega^e \) and \( E := \{ n \in \mathbb{N} : f(\ell^\beta n) = \vec{x}, g(k^\alpha n) = \vec{y}\} \). Suppose that \( \bar{d}(E) > 0 \). Then \( x_\phi((m)_k) = y_\psi((m)_\ell) \) for all \( 0 \leq m < \min(k^\alpha, \ell^\beta) \).

**Proof of Lemma:**

- \( f(k^\alpha \ell^\beta n + m) = f_\phi((m)_k)(\ell^\beta n) = x_\phi((m)_k) \) for almost all \( n \in E \).

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- Since \( \bar{d}(E) > 0 \), there is at least one \( n \in \mathbb{N} \) such that

\[
x_\phi((m)_k) = f(k^\alpha \ell^\beta n + m) = y_\psi((m)_\ell).
\]
Proof of asymptotic Cobham’s theorem

Lemma

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$ and $\vec{y} \in \Omega^e$. Suppose that

$$\bar{d} \left( \left\{ n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x}, \vec{g}(k^\alpha n) = \vec{y} \right\} \right) > 0. \quad (\ast)$$

Then $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ for all $0 \leq m < \min(k^\alpha, \ell^\beta)$.

Corollary

Let $\vec{x} \in \Omega^d$. The sequence $x_{\phi((m)_k)}$ is eventually periodic, provided that (\ast) holds for arbitrarily large $\alpha, \beta \in \mathbb{N}$ for some $\vec{y} \in \Omega^e$. Call such $\vec{x}$ “good”.

- Directly by definition, $x_{\phi((m)_k)}$ is $k$-automatic and $y_{\psi((m)_\ell)}$ is $\ell$-automatic.
- By Lemma, $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ is $k$- and $\ell$-automatic.
- By Cobham’s theorem, $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ is eventually periodic.

Let $q$ be the least common multiple of periods from Corollary above. For ease of notation assume $x_{\phi((m)_k)}$ is genuinely periodic.
Proof of asymptotic Cobham’s theorem

**Lemma**

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$ and $\vec{y} \in \Omega^e$. Suppose that

\[ \bar{d}\left( \{ n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x}, \vec{g}(k^\alpha n) = \vec{y} \} \right) > 0. \]  

(\*)

Then $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ for all $0 \leq m < \min(k^\alpha, \ell^\beta)$.

**Corollary**

Let $\vec{x} \in \Omega^d$. The sequence $x_{\phi((m)_k)}$ is eventually periodic, provided that (\*) holds for arbitrarily large $\alpha, \beta \in \mathbb{N}$ for some $\vec{y} \in \Omega^e$. Call such $\vec{x}$ “good”.

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Lemma

Let $\alpha, \beta \in \mathbb{N}$, $\vec{x} \in \Omega^d$ and $\vec{y} \in \Omega^e$. Suppose that

$$d \left( \left\{ n \in \mathbb{N} : \vec{f}(\ell^\beta n) = \vec{x}, \vec{g}(k^\alpha n) = \vec{y} \right\} \right) > 0.$$  (\ast)

Then $x_{(m)_k} = y_{(m)_\ell}$ for all $0 \leq m < \min(k^\alpha, \ell^\beta)$.

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\]

Then \( x_{\phi((m)_k)} = y_{\psi((m)_\ell)} \) for all \( 0 \leq m < \min(k^\alpha, \ell^\beta) \).

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Let \( \vec{x} \in \Omega^d \). The sequence \( x_{\phi((m)_k)} \) is eventually periodic, provided that (*) holds for arbitrarily large \( \alpha, \beta \in \mathbb{N} \) for some \( \vec{y} \in \Omega^e \). Call such \( \vec{x} \) “good”.

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Then $x_{\phi((m)_k)} = y_{\psi((m)_\ell)}$ for all $0 \leq m < \min(k^\alpha, \ell^\beta)$.

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Let $q$ be the least common multiple of periods from Corollary above. For ease of notation assume $x_{\phi((m)_k)}$ is genuinely periodic.
Proof of asymptotic Cobham’s theorem

**Corollary**

The sequence $x_{\phi((m)_k)}$ has period $q$ for each “good” $\vec{x} \in \Omega^d$.

**Lemma**

Let $n \in \mathbb{N}$. Then

$$f(n + q) = f(n),$$

provided that there exists a decomposition $n = k^\alpha n' + m$ where $m < k^\alpha - q$ and $\vec{x} := \vec{f}(n')$ is “good”.

**Proof:**

$$f(k^\alpha n' + m + q) = x_{\phi((m+q)_k)} = x_{\phi((m)_k)} = f(k^\alpha n' + m).$$

**Lemma**

For asymptotically almost all $n$, there exists a decomposition $n = k^\alpha n' + m$ where $n', m, \alpha \in \mathbb{N}, m < k^\alpha - q$, $\vec{f}(n')$ is “good”.

**Proof idea:** For each $\alpha < \log_k n$, there is a positive chance to find the decomposition.

**Corollary**

The sequence $f(n)$ is asymptotically invariant under shift by $q$, QED.
Proof of asymptotic Cobham’s theorem

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Let \( n \in \mathbb{N} \). Then

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Proof: \( f(k^\alpha n' + m + q) = x_{\phi((m+q)_k)} = x_{\phi((m)_k)} = f(k^\alpha n' + m) \).

Lemma

For asymptotically almost all \( n \), there exists a decomposition \( n = k^\alpha n' + m \) where \( n', m, \alpha \in \mathbb{N}, m < k^\alpha - q \), \( \vec{f}(n') \) is “good”.

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The sequence \( f(n) \) is asymptotically invariant under shift by \( q \), QED.
Proof of asymptotic Cobham’s theorem

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The sequence $x_{\phi((m)_{k})}$ has period $q$ for each “good” $\vec{x} \in \Omega^d$.

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For asymptotically almost all $n$, there exists a decomposition $n = k^\alpha n' + m$ where $n', m, \alpha \in \mathbb{N}$, $m < k^\alpha - q$, $\vec{f}(n')$ is “good”.

**Proof idea:** For each $\alpha < \log_k n$, there is a positive chance to find the decomposition.

**Corollary**

The sequence $f(n)$ is asymptotically invariant under shift by $q$, QED.
Proof of “mixed” Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f : \mathbb{N} \to \Omega$ is $k$-automatic and asymptotically $\ell$-automatic;
- By previous theorem, $f$ is asymptotically invariant under shift by some $q \geq 1$;
- To simplify, assume that $q = 1$.

Lemma

Let $g : \mathbb{N} \to \{0, 1\}$ be a $k$-automatic sequence with $g(n) \simeq 0$. Then there is $n_0 \in \mathbb{N}$ with

$$g(k^\alpha n_0 + m) = 0 \quad \text{for all } \alpha \in \mathbb{N}, \ 0 \leq m < k^\alpha.$$

Proof idea: Pick an automaton computing $g$, reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick $v \in \Sigma_k^*$ such that $\delta(s_0, v)$ lies in a strongly connected component, and put $n_0 = [v]_k$.

- Let $n_0$ be the constant from the Lemma applied to the $k$-automatic sequence $g(n) = \begin{cases} 1 & \text{if } f(n + 1) \neq f(n), \\ 0 & \text{if } f(n + 1) = f(n). \end{cases}$
Proof of “mixed” Cobham’s theorem

Assumptions and notation:

- \( k, \ell \geq 2 \) are multiplicatively independent integers;
- \( f : \mathbb{N} \to \Omega \) is \( k \)-automatic and asymptotically \( \ell \)-automatic;
- By previous theorem, \( f \) is asymptotically invariant under shift by some \( q \geq 1 \);
- To simplify, assume that \( q = 1 \).

Lemma

Let \( g : \mathbb{N} \to \{0, 1\} \) be a \( k \)-automatic sequence with \( g(n) \simeq 0 \). Then there is \( n_0 \in \mathbb{N} \) with

\[
g(k^\alpha n_0 + m) = 0 \quad \text{for all } \alpha \in \mathbb{N}, \ 0 \leq m < k^\alpha.
\]

Proof idea: Pick an automaton computing \( g \), reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick \( v \in \Sigma_k^* \) such that \( \delta(s_0, v) \) lies in a strongly connected component, and put \( n_0 = [v]_k \).

- Let \( n_0 \) be the constant from the Lemma applied to the \( k \)-automatic sequence

\[
g(n) = \begin{cases} 1 & \text{if } f(n + 1) \neq f(n), \\ 0 & \text{if } f(n + 1) = f(n). \end{cases}
\]
Proof of “mixed” Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f : \mathbb{N} \rightarrow \Omega$ is $k$-automatic and asymptotically $\ell$-automatic;
- By previous theorem, $f$ is asymptotically invariant under shift by some $q \geq 1$;
- To simplify, assume that $q = 1$.

Lemma

Let $g : \mathbb{N} \rightarrow \{0, 1\}$ be a $k$-automatic sequence with $g(n) \simeq 0$. Then there is $n_0 \in \mathbb{N}$ with

$$g(k^\alpha n_0 + m) = 0 \quad \text{for all } \alpha \in \mathbb{N}, \ 0 \leq m < k^\alpha.$$

Proof idea: Pick an automaton computing $g$, reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick $v \in \Sigma_k^*$ such that $\delta(s_0, v)$ lies in a strongly connected component, and put $n_0 = [v]_k$.

- Let $n_0$ be the constant from the Lemma applied to the $k$-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n+1) \neq f(n), \\ 0 & \text{if } f(n+1) = f(n). \end{cases}$$
Proof of “mixed” Cobham’s theorem

Assumptions and notation:

- $k, \ell \geq 2$ are multiplicatively independent integers;
- $f : \mathbb{N} \to \Omega$ is $k$-automatic and asymptotically $\ell$-automatic;
- By previous theorem, $f$ is asymptotically invariant under shift by some $q \geq 1$;
- To simplify, assume that $q = 1$.

Lemma

Let $g : \mathbb{N} \to \{0, 1\}$ be a $k$-automatic sequence with $g(n) \simeq 0$. Then there is $n_0 \in \mathbb{N}$ with

$$g(k^\alpha n_0 + m) = 0 \quad \text{for all } \alpha \in \mathbb{N}, \ 0 \leq m < k^\alpha.$$

Proof idea: Pick an automaton computing $g$, reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick $v \in \Sigma_k^*$ such that $\delta(s_0, v)$ lies in a strongly connected component, and put $n_0 = [v]_k$.

Let $n_0$ be the constant from the Lemma applied to the $k$-automatic sequence

$$g(n) = \begin{cases} 
1 & \text{if } f(n + 1) \neq f(n), \\
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\end{cases}$$
Proof of “mixed” Cobham’s theorem

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**Lemma**

Let $g : \mathbb{N} \to \{0, 1\}$ be a $k$-automatic sequence with $g(n) \simeq 0$. Then there is $n_0 \in \mathbb{N}$ with

$$g(k^\alpha n_0 + m) = 0 \quad \text{for all } \alpha \in \mathbb{N}, \ 0 \leq m < k^\alpha.$$ 

Proof idea: Pick an automaton computing $g$, reading input from the most significant digit. The output function is 0 on each strongly connected component. Pick $v \in \Sigma_k^*$ such that $\delta(s_0, v)$ lies in a strongly connected component, and put $n_0 = [v]_k$.

Let $n_0$ be the constant from the Lemma applied to the $k$-automatic sequence

$$g(n) = \begin{cases} 1 & \text{if } f(n + 1) \neq f(n), \\ 0 & \text{if } f(n + 1) = f(n). \end{cases}$$
Proof of “mixed” Cobham’s theorem

**Assumptions and notation:**

- $k, \ell \geq 2$ are multiplicatively independent integers;
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Frequencies

**Definition**

Let \( f : \mathbb{N} \to \Omega, \omega \in \Omega \). The \((\text{asymptotic} / \text{logarithmic})\) frequency of \( \omega \) if \( f \) is:

\[
\text{freq}(f; \omega) := \lim_{N \to \infty} \frac{1}{N} \cdot \# \{ n < N : f(n) = \omega \},
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\text{freq}_{\log}(f; \omega) := \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=0}^{N-1} \frac{1_{\omega}(n)}{n + 1}.
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**Proposition (Frequencies of symbols in automatic sequences)**

Let \( f : \mathbb{N} \to \Omega \) be automatic and \( \omega \in \Omega \). Then

- the logarithmic frequency \( \text{freq}_{\log}(f; \omega) \) exists;
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There exists an asymptotically 2-automatic sequence $f : \mathbb{N} \to \{0, 1\}$ such that

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For each $\theta \in [0, 1]$ there exists an asymptotically 2-automatic sequence $f : \mathbb{N} \to \{0, 1\}$ such that $\text{freq}(f; 1) = \theta$. 
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Frequencies — Proof ideas

- We can write the binary expansion of any \( n \in \mathbb{N} \) as
  \[
  (n)_2 = u_1^{(n)} u_2^{(n)} \cdots u_{r(n)}^{(n)} v^{(n)},
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Classification problems

**General questions:** Fix the base $k \geq 2$.

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- Multiplicative sequences are defined in terms of the multiplicative structure of \( \mathbb{N} \).
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The following families of sequences are automatic and multiplicative:

- Dirichlet characters, and more generally periodic multiplicative sequences;
- $f(n) = \omega^{\nu_p(n)}$, where $\nu_p(n) = \max \{ \nu : p^\nu | n \}$ and $\omega = \exp(2\pi i/r)$;
- eventually zero multiplicative sequences.

### Example (Automatic multiplicative semigroups)

The following families of multiplicative semigroups are automatic:

- periodic semigroups;
- $\{n \in \mathbb{N} : \nu_p(n) \equiv 0 \text{ mod } r\}$;
- $\mathbb{N} \setminus \{p^\alpha : \alpha \in \mathbb{N}\}$;
- $mX \cup m^2\mathbb{N}$ where $m \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ is any automatic set.
Classification of automatic multiplicative sequences

**General fact**

Fix a prime $p$. Each non-zero multiplicative sequence $f$ has a unique representation

$$f(n) = h(\nu_p(n)) \cdot g(n/p^{\nu_p(n)}),$$

(†) where $h(0) = 1$ and $g(pn) = 0$ for all $n$. Additionally, $g$ is multiplicative.

**Theorem (K., Lemańczyk, Müllner 2020)**

Fix $k \geq 2$ and let $f : \mathbb{N} \to \mathbb{C}$ be a non-zero multiplicative sequence.

- If $k$ is a power of a prime $p$ then $f$ is $k$-automatic iff $h$ and $g$ given by (†) are eventually periodic. (In this case, $g$ must be either periodic or eventually zero.)
- If $k$ has $\geq 2$ prime divisors then $f$ is $k$-automatic iff $f$ is eventually periodic.

**Remark:** Conversely, each sequence $f$ of the form described above is both $k$-automatic and multiplicative.
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- **If** $f(q) \neq 0$ and $\phi(qm) = \phi(qm')$ then $\phi(m) = \phi(m')$:

\[
 f_\phi(m)(n) \simeq f(k^i n + m) = f(q)^{-1} f(k^i qn + qm) \simeq f(q)^{-1} f_\phi(qm)(qn)
\]

\[
 \simeq f(q)^{-1} f_\phi(qm')(qn) = \cdots = f_\phi(m')(n).
\]

- For each $q \in \mathbb{N}$ without small prime factors, there exists $\hat{q} \in \mathbb{N}$ such that $\phi(\hat{q}q) = \phi(1)$.
- The last two items imply that if $\phi(q) = \phi(q')$ and $\phi(r) = \phi(r')$ then $\phi(qr) = \phi(q'r')$.
- Define a semigroup operation $\odot$ on (a subset of) $\Sigma_d$ by $\phi(q) \odot \phi(r) = \phi(qr)$.
- Apply classification of automatic multiplicative sequences to conclude that $\phi$ is periodic (on integers without small prime factors).
- Periodicity of $\phi$ implies asymptotic periodicity of $f$.
- Combining the last item with the fact from [Klurman 2017] finishes the argument.
Classification of automatic multiplicative sequences

Theorem (K.)

Fix $k \geq 2$ and let $f : \mathbb{N} \to \mathbb{C}$ be an asymptotically automatic multiplicative sequence. Then there exists $\chi : \mathbb{N} \to \mathbb{C}$ that is either a Dirichlet character or identically 0, such that $f(p^\alpha) = \chi(p^\alpha)$ for all sufficiently large primes $p$ and all $\alpha \in \mathbb{N}$.

Proof ideas:

- Key ingredient [Klurman 2017]: If $f$ is finitely-valued, multiplicative and asymptotically invariant under a shift then $f \simeq 0$ or $f$ is periodic.
- We can use old tricks to assume that $f$ is completely multiplicative.
- Like earlier, we can find $f_0, f_1, \ldots, f_{d-1}$ and $k$-automatic $\phi : \Sigma_k^* \to \Sigma_d$ such that $f(k^{|u|}n + [u]_k) \simeq f_{\phi(u)}(n)$ for $u \in \Sigma_k^*$. To simplify, assume that $\phi(0u) = \phi(u)$.
- If $f(q) \neq 0$ and $\phi(qm) = \phi(qm')$ then $\phi(m) = \phi(m')$:
  
  \[ f_{\phi(m)}(n) \simeq f(k^n m) = f(q)^{-1} f(k^n qn + qm) \simeq f(q)^{-1} f_{\phi(qm)}(qn) \]
  
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- Periodicity of $\phi$ implies asymptotic periodicity of $f$.
- Combining the last item with the fact from [Klurman 2017] finishes the argument.
Thank you for your attention!
**Automatic semigroups**

**General fact**
Let $p$ be a prime and let $E$ be a $p$-automatic set. Then $E$ can be decomposed as
\[ E = E_0 \cup pE_1 \cup p^2E_2 \cup \ldots, \]
the sequence $E_0, E_1, E_2, \ldots$ is eventually periodic, and $p \nmid n$ for all $n \in E_i$.

**Theorem (Klurman, K. 2023+)**
Let $k \geq 2$ and let $E \subseteq \mathbb{N}$ be a $k$-automatic semigroup. Assume further that $E$ contains an infinite pairwise coprime subset.

- If $k$ is a power of a prime $p$ then for each $i \geq 0$, the sets $E_i$ are asymptotically periodic.
- If $k$ has $\geq 2$ prime divisors then $E$ is asymptotically periodic.

**Recall:** When all elements of $E$ are allowed to share a factor, we get examples of the type $E = mX \cup m^2\mathbb{N}$, so the assumption cannot be removed. Not all sets of the above form are semigroups, but specifying which are is more mundane than difficult.
Automatic semigroups

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Automatic multiplicatively stable sets

**Definition**
- For $E \subset \mathbb{N}$ and $m \in \mathbb{N}$ we let $E/m = \{n \in \mathbb{N} : mn \in E\}$. Note: $(mE)/m = E$.

**Proposition**
Let $E \subset \mathbb{N}$ be a $k$-automatic set. There exists a constant $\Delta \in \mathbb{N}$ with the property that for each $q \in \mathbb{N}$ with $\gcd(q,k) = 1$ and each $c \in \mathbb{Z}$, if we put $q' := \gcd(q,\Delta)$ then

$$d_{\log}((E-c)/q) = d_{\log}((E-c)/q').$$

**Observation**
Let $E \subset \mathbb{N}$ be a $k$-automatic semigroup, $q \in E$ and $\gcd(q,k\Delta) = 1$. Then $E/q \cong E$.

**Proof:**
- Since $E$ is a semigroup and $q \in E$, we have $E/q \supseteq E$.
- Since $\gcd(q,\Delta) = 1$, we have $d_{\log}(E/q) = d_{\log}(E)$.
- Combining the two points above: $d_{\log}(E/q\Delta E) = d_{\log}(E/q) - d_{\log}(E) = 0$. □
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Let $E \subseteq \mathbb{N}$ be a $k$-automatic semigroup, $q \in E$ and gcd$(q, k\Delta) = 1$. Then $E/q \simeq E$.

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Multiplicative invariance

Definition
Let $E \subset \mathbb{N}$ be a set. We define the asymptotically invariant and reversible sets:

\[
\text{Inv}(E) := \{ q \in \mathbb{N} : E/q \simeq E \},
\]
\[
\text{Rev}(E) := \{ q \in \mathbb{N} : q\mathbb{N} \cap \text{Inv}(E) \neq \emptyset \}.
\]

Theorem (Klurman, K. 2023+)
Let $k \geq 2$, let $E, F \subset \mathbb{N}$ be $k$-automatic sets with $F \subset \text{Inv}(E)$ and $d_{\log}(F') > 0$.

- If $k$ is a power of a prime $p$ then $E = E_0 \cup pE_1 \cup p^2E_2 \cup \ldots$, where $E_i$ are asymptotically periodic.
- If $k$ has $\geq 2$ prime divisors then $E$ is asymptotically periodic.

Proof ideas (slightly oversimplified)
- The set Rev($E$) is periodic.
- We can construct a finite group $G_E := \text{Rev}(E)/\text{Inv}(E)$.
- The quotient map $\pi_E : \mathbb{N} \to G_E \cup \{0\}$ is $k$-automatic.
- The map $\pi_E$ is periodic (by classification of automatic multiplicative sequences).
- The set $E$ is asymptotically periodic.
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**Theorem (Klurman, K. 2023+)**

Let $k \geq 2$, let $E, F \subset \mathbb{N}$ be $k$-automatic sets with $F \subset \text{Inv}(E)$ and $d_{\text{log}}(F) > 0$.

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Asymptotically automatic sequences

Question

- Can we characterise pairs of $k$-automatic sets $E, F \subset \mathbb{N}$ with $F \subset \text{Inv}(E)$?
- Can we use assumptions like $E/q \simeq E$ or $E/q \supseteq E$ when $q$ is not coprime to $k$?

Example

Let $E$ be 10-automatic set with $2 \in \text{Inv}(E)$. Then $1_E$ is asymptotically 5-automatic;

$$1_E(5^\alpha n + m) \simeq 1_E(10^\alpha n + 2^\alpha m) \in \mathcal{N}_{10}(1_E(n))$$

for each $\alpha, m \in \mathbb{N}$ with $m < 5^\alpha$, and hence $\#(\mathcal{N}_5(1_E)/\simeq) \leq \#(\mathcal{N}_{10}(1_E))$.

Corollary

If $E \subset \mathbb{N}$ is a 10-automatic set with $2 \in \text{Inv}(E)$ then $E$ is asymptotically periodic.
Asymptotically automatic sequences

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