α-Hurwitz continued fractions

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Complex continued fractions

In 1887 A. Hurwitz introduced an algorithm producing continued fractions

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \quad a_n \in \mathbb{Z}[i],$$

for complex numbers z in the set

$$U = \left\{ z = x + yi \in \mathbb{C} \,:\, x, y \in \left[-\frac{1}{2}, \frac{1}{2} \right) \right\}.$$

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These are generated by the transformation

$$T(z) = \frac{1}{z} - \left\lfloor \frac{1}{z} \right\rfloor_H = \frac{1}{z} - a_1(z),$$

where $\lfloor \frac{1}{z} \rfloor_H$ is the Gaussian integer that translates $\frac{1}{z}$ to U.

Hurwitz' algorithm

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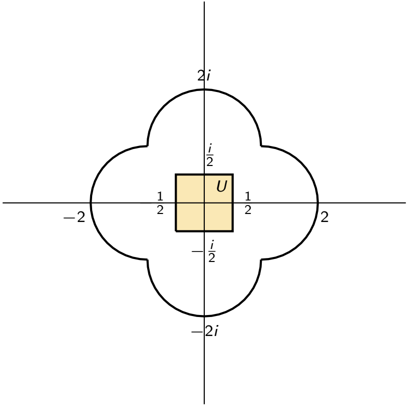

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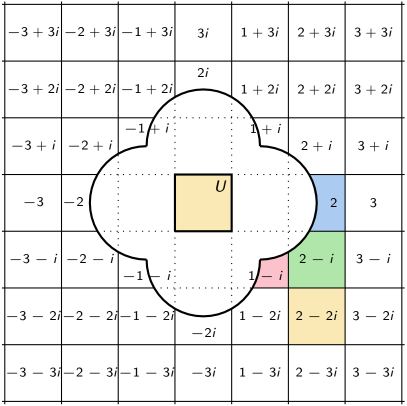

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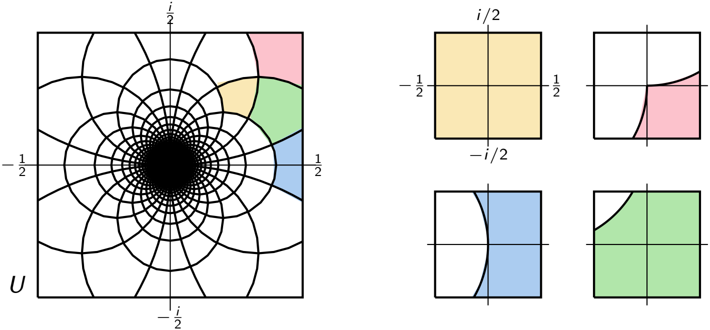

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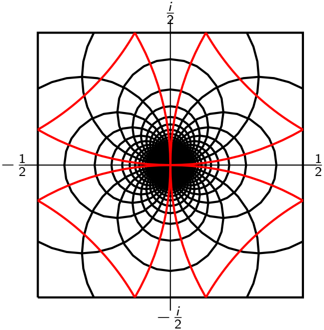
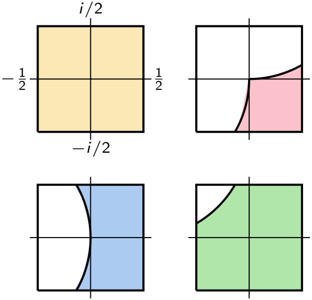
Markov partition





$$T(z) = \frac{1}{z} - \left\lfloor \frac{1}{z} \right\rfloor_H$$

All boundaries (and images of boundaries) are contained in the circles $C_1(z)$ with $z \in \mathbb{Z}[i]$.

Results by Hurwitz I

$T(z) = \frac{1}{z} - \lfloor \frac{1}{z} \rfloor_H = \frac{1}{z} - a_1(z)$.

For each $n \geq 1$ set $a_n(z) = \lfloor \frac{1}{T^{n-1}(z)} \rfloor_H$. Then

$$z = \frac{1}{a_1 + T(z)} = \frac{1}{a_1 + \frac{1}{a_2 + T^2(z)}} = \cdots = \frac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{a_n + T^n(z)}}}.$$

Hurwitz (1887):

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- This process converges and gives an infinite continued fraction expansion for each z such that $T^n(z) \neq 0$ for all n.

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- Characterisation of the periodic expansions.

Results by Hurwitz II

For

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

set for each $n \geq 1$,

$$\frac{P_n}{Q_n} = \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_n}} \in \mathbb{Q}(i).$$

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$$\left| z - \frac{P_n}{Q_n} \right| < \frac{1}{B|Q_n|^2}.$$

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Perron (1913): $B \geq \sqrt{2} - 1$.

Lakein (1973): $B = 1$ and this is best possible.

Dynamical results on the Hurwitz map

Nakada (1976): Existence and ergodicity of an absolutely continuous invariant probability measure μ. (See also **Hensley (2006)**, adding mixing.)

Ito (1987): Description of natural extension domain.

Schweiger (2000, 2011): Kuzmin's Theorem and density function is piecewise Lipschitz.

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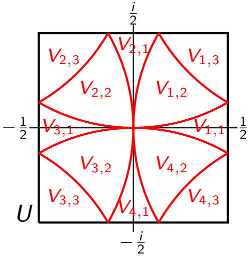
Ei, Ito, Nakada, Natsui (2019): Construction of natural extension.

$$V^*_{k,l} = \overline{\left\{ -\frac{Q_n(z)}{Q_{n-1}(z)} \; : \; z \in U, \; T^n(z) \in V_{k,l}, \; n \geq 1 \right\}},$$

$$\hat{U} = \bigcup_{k,l=1}^{4} V_{k,l} \times V^*_{k,l},$$

$$\hat{T} : \hat{U} \to \hat{U}; \; (z,w) \mapsto \left(\frac{1}{z} - a_1(z), \frac{1}{w} - a_1(z) \right),$$

$$\hat{\mu}(A) = \frac{1}{C} \int_A \frac{1}{|z-w|^4} \, d\text{Leb}(z,w).$$



Natural extension

Let $\pi : \hat{U} \to U$ be the projection onto the first coordinate. Let $\hat{B}$ and $\mathcal{B}$ be the Borel σ-algebras on $\hat{U}$ and U, respectively.

Ei, Ito, Nakada, Natsui (2019):
The collection $(\hat{U}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T})$ is the **natural extension** of the system $(U, \mathcal{B}, \mu = \hat{\mu} \circ \pi^{-1}, T)$:

- $0 < \hat{\mu}(\hat{U}) < \infty$,
- $\hat{T}$ is almost everywhere invertible,
- π is measurable and almost everywhere surjective,
- $\pi \circ \hat{T} = T \circ \pi$,
- $\hat{\mathcal{B}} = \bigvee_{k \geq 0} \hat{T}^k \circ \pi^{-1} \mathcal{B}$.

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This makes μ, given for $A \subseteq V_{k,l}$ by

$$\mu(A) = \frac{1}{C} \int_{V^*_{k,l}} \frac{1}{|z - w|^4} \, d\text{Leb}(w),$$

the unique absolutely continuous invariant measure for T and ergodic.

Legendre constant and other important ingredients

Ei, Ito, Nakada, Natsui (2019):

Important ingredients in the proof of this result:

- Symmetry of the map.
- Markov partition;
- $|Q_n| > |Q_{n-1}|$ for all n;
- Legendre constant: There exists a constant $L > 0$ such that

$$\left| z - \frac{P}{Q} \right| < L \cdot \frac{1}{|Q|^2}$$

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Dani and Nogueira (2014): Periodic expansions, growth of $|Q_n|$ in more general setting.

Kirsebom (2021): Extreme value theory on size of digits.

González Robert (2022): Borel-Bernstein Theorem.

Hiary and Vandehey (2022): Computational results on invariant density of μ.

Ei, Nakada and Natsui (2022): Simplified proof for Legendre constant.

Abrams (2022): Description of natural extension domain for various complex continued fractions.

Definition of α-Hurwitz systems

We introduce a **shifted version of the Hurwitz system**:

For $\alpha = (\alpha_1, \alpha_2)$ let

$$U_\alpha = \{z = x + yi \in \mathbb{C} \,:\, x \in [\alpha_1 - 1, \alpha_1)\,,\, y \in [\alpha_2 - 1, \alpha_2)\}$$

and

$$T_\alpha(z) = \frac{1}{z} - \left\lfloor \frac{1}{z} \right\rfloor_\alpha = \frac{1}{z} - a_1^\alpha(z),$$

where $\lfloor \frac{1}{z} \rfloor_\alpha$ is the Gaussian integer that translates $\frac{1}{z}$ to U_α.

α-Hurwitz map

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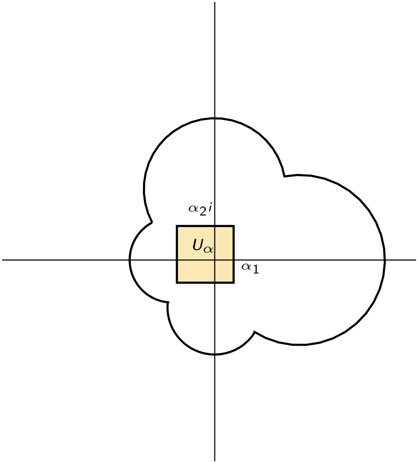

$\alpha = (1/3, 3/5)$

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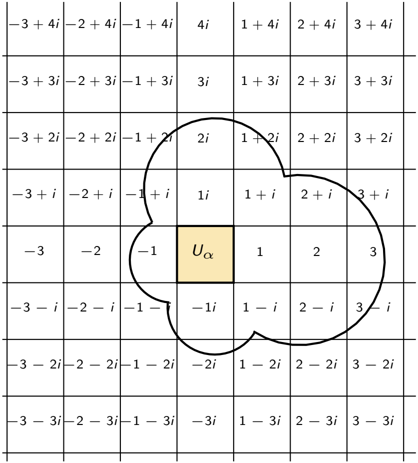

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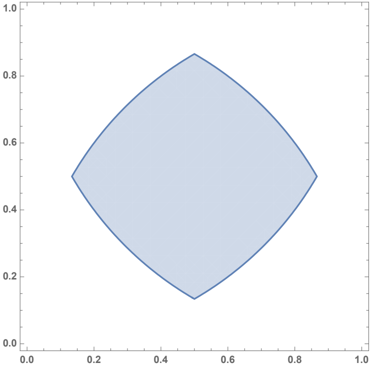
Convergence of continued fractions

Continued fraction expansions obtained from this map converge if

$$\alpha \in \mathcal{D} := B_1(0,0) \cap \overline{B_1(1,0)} \cap \overline{B_1(0,1)} \cap \overline{B_1(1,1)}.$$

(See e.g. Dani and Nogueira (2014).)



Important ingredients

- Symmetry of the map $\rightarrow$ lost.
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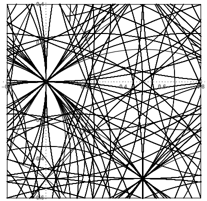
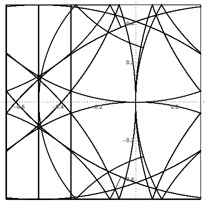
Markov partitions

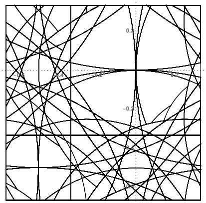
Numerical results show that Markov partitions exist for other values of α.



$\alpha = (\frac{4}{5}, \frac{2}{5})$



$\alpha = (\frac{1}{3}, \frac{1}{2})$



$\alpha = (\frac{1}{3}, \frac{1}{3})$

Markov partitions proof I

Theorem

The map T_α admits a Markov partition for each $\alpha \in \mathbb{Q}^2 \cap \mathcal{D}$.

Proof:

Let $\alpha = (\frac{p}{q}, \frac{r}{s})$.

Under the map $z \mapsto \frac{1}{z}$ the boundaries of U_α become arcs in the circles

$$C_{\frac{q}{2(q-p)}}\left(\frac{q}{2(p-q)}\right), \ C_{\frac{q}{2p}}\left(\frac{q}{2p}\right), \ C_{\frac{s}{2(s-r)}}\left(\frac{-si}{2(r-s)}\right), \ C_{\frac{s}{2r}}\left(\frac{-si}{2r}\right).$$

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Each of these circles satisfies an equation of the form

$$2A_0 z\overline{z} - \overline{B_0}z - B_0\overline{z} = 0, \quad A_0 \in \mathbb{Z},\ B_0 \in \mathbb{Z}[i].$$

Markov partitions proof II

Under T_α each of the boundary lines are first inverted by the map $z \mapsto \frac{1}{z}$ and then translated by a Gaussian integer so that the result intersects U_α.

For each of the circles we inductively define a sequence of generalised circles (C_n) as follows.

Set

$$C_0 = \{z \in \mathbb{C} \,:\, 2A_0 z\overline{z} - \overline{B_0}z - B_0\overline{z} = 0\}.$$

If C_n is defined, let $z_n \in \mathbb{Z}[i]$ be any Gaussian integer such that $C_n - z_n \cap U_\alpha \neq \emptyset$ and set $C_{n+1} = \frac{1}{C_n - z_n}$.

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With induction one can show that

$$C_n = \{z \in \mathbb{C} : 2A_n z\overline{z} - \overline{B_n} z - B_n \overline{z} + 2A_{n-1} = 0\},$$

where

$$2A_{n+1} = 2A_n z_n \overline{z}_n - \overline{B}_n z_n - B_n \overline{z}_n + 2A_{n-1},$$
$$B_{n+1} = \overline{B}_n - 2A_n \overline{z}_n.$$

Markov partitions proof III

One can deduce by induction that $B_n\overline{B}_n - 4A_{n-1}A_n = B_0^2$.

In case C_n is a circle, this gives that the radius ρ_n satisfies

$$\rho_n^2 = \frac{B_0^2}{4A_n^2},$$

which immediately implies $\rho_n < \frac{|B_0|}{2}$.

Markov partitions proof III

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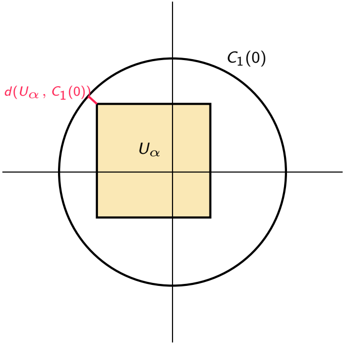
Claim: $\rho_n > d(U_\alpha, C_1(0))/2 =: \rho_{\min}$.



Markov partitions proof IV

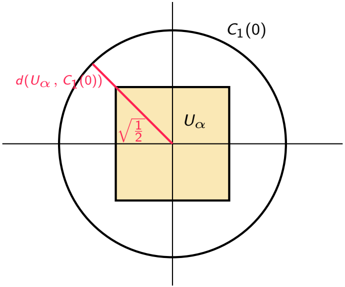
Claim: $\rho_n > d(U_\alpha, C_1(0))/2 =: \rho_{\min}$.

For C_0 this is true: C_0 is one of

$$C_{\frac{q}{2(q-p)}}\left(\frac{q}{2(p-q)}\right),\ C_{\frac{q}{2p}}\left(\frac{q}{2p}\right),\ C_{\frac{s}{2(s-r)}}\left(\frac{-si}{2(r-s)}\right),\ C_{\frac{s}{2r}}\left(\frac{-si}{2r}\right),$$

so $\rho_0 \in \left\{\frac{q}{2(q-p)}, \frac{q}{2p}, \frac{s}{2(s-r)}, \frac{s}{2r}\right\}$. Hence,

$$\rho_0 > \frac{1}{2} > 1 - \sqrt{\frac{1}{2}} \geq \frac{d(U_\alpha, C_1(0))}{2} = \rho_{\min}.$$

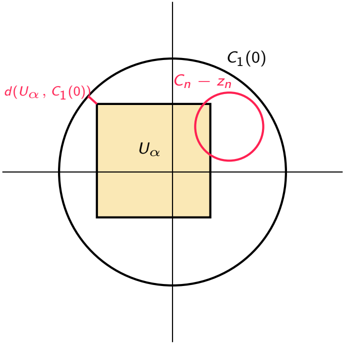


Markov partitions proof V

Claim: $\rho_n > d(U_\alpha, C_1(0))/2 =: \rho_{\min}$.

Suppose that C_{n+1} is a circle and the statement is true for all $0 \leq k \leq n$ for which C_k is a circle.

If $C_n - z_n$ lies completely within $C_1(0)$, then C_n is a circle and $\rho_{n+1} > \rho_n > \rho_{\min}$ by the induction hypothesis.

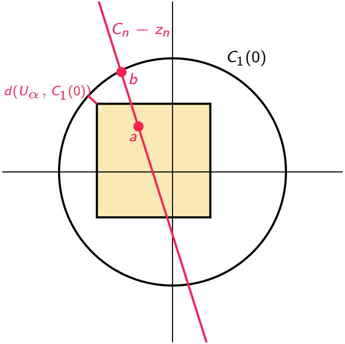


Markov partitions proof VI

If $C_n - z_n$ does not lie completely within $C_1(0)$, then there is a point $a \in C_n - z_n \cap U_\alpha$ and a point $b \in C_n - z_n \cap C_1(0)$.

Then $|a| < 1 - 2\rho_{\min}$ and $|\frac{1}{a}| > \frac{1}{1-2\rho_{\min}} > 1 + 2\rho_{\min}$.

Since $|\frac{1}{b}| = 1$, then $d(\frac{1}{a}, \frac{1}{b}) > 2\rho_{\min}$ and $\frac{1}{a}, \frac{1}{b} \in C_{n+1}$.



Markov partitions proof VII

Each circle in (C_n) has a radius $\rho_{\min} < \rho_n = \sqrt{\frac{|B_0|}{4A_n^2}} < \frac{|B_0|}{2}$.

Each generalised circle satisfies an equation

$$2A_n z\overline{z} - \overline{B_n} z - B_n \overline{z} + 2A_{n-1} = 0$$

with $B_n \overline{B}_n - 4A_{n-1}A_n = B_0^2$.

This implies that only finitely many values of A_n and B_n occur, where the bounds only depend on α.

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This implies that only finitely many values of A_n and B_n occur, where the bounds only depend on α.

Hence, only finitely many generalised circles can appear in any of the sequences (C_n).

This gives a Markov partition of the map T. $\square$

Important ingredients

- Symmetry of the map $\rightarrow$ lost.
- Markov partition $\rightarrow$ for $\alpha \in \mathbb{Q}^2 \cap \mathcal{D}$.
- $|Q_n| > |Q_{n-1}|$ for all $n \rightarrow$?
- Legendre constant: There exists a constant $L > 0$ such that

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implies that $\frac{P}{Q} = \frac{P_n}{Q_n}$ for some $n \geq 0 \rightarrow$?

Increasing $|Q_n|$

Essential to the proof given by Hurwitz that the sequence $(|Q_n|)_{n\geq 1}$ is increasing is the following table of non-admissible digits:

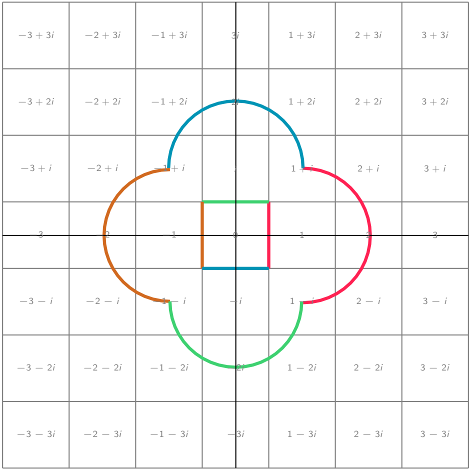
Tabelle unmöglicher Zahlfolgen.

	a_n	a_{n+1}	a_{n+2}	a_{n+3}
I.	$-2, 2i, -1+i, -2+i, -1+2i$	$1+i$		
II.	$2, 2i, 1+i$	$-2+2i$		
III.	$2+i, 1+2i$	$-2+2i$	$1+i$	
IV.	$-2, 2i, -1+i$	$2+2i$		
V.	$-2+i, -1+2i$	$2+2i$	$-2+2i$	$1+i$

There are corresponding tables for $1-i$, $-1+i$, $-1-i$.

Small perturbations I

We considered small perturbations of α around $(\frac{1}{2}, \frac{1}{2})$. Here: $(0.495, 0.505)$.



Small perturbations II

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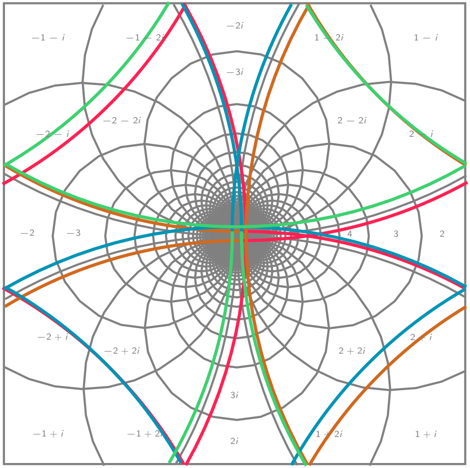
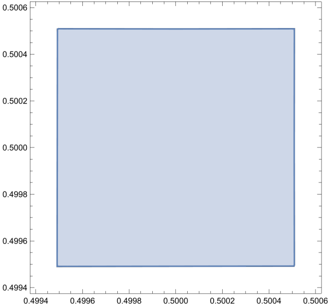


Region $\mathcal{D}'$

Theorem

There is a small neighborhood $\mathcal{D}'$ of $(\frac{1}{2}, \frac{1}{2})$, such that for any $\alpha \in \mathcal{D}'$ the table of admissible words from Hurwitz still holds.

The sequence $(|Q_n|)_{n\geq 1}$ is increasing if and only if $\alpha \in \mathcal{D}'$.



Inequalities

For $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \leq \frac{1}{2}$ it holds that $\alpha \in \mathcal{D}'$ if and only if

$$\left(\frac{\alpha_1 - 2}{(\alpha_1 - 2)^2 + (\alpha_2 + 1)^2} - \frac{9\alpha_2 - 5}{29\alpha_2 - 15} + 2\right)^2 + \left(\frac{-(\alpha_2 + 1)}{(\alpha_1 - 2)^2 + (\alpha_2 + 1)^2} + \frac{9 - 16\alpha_2}{58\alpha_2 - 30} + 2\right)^2 < \frac{1}{(58\alpha_2 - 30)^2},$$

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For $\alpha_1 \leq \frac{1}{2}, \alpha_2 > \frac{1}{2}$ and $\alpha_1 > \frac{1}{2}, \alpha_2 \leq \frac{1}{2}$ and $\alpha_1, \alpha_2 > \frac{1}{2}$ there are similar inequalities.

Important ingredients

- Symmetry of the map $\rightarrow$ lost.
- Markov partition $\rightarrow$ for $\alpha \in \mathbb{Q}^2 \cap \mathcal{D}$.
- $|Q_n| > |Q_{n-1}|$ for all $n \rightarrow$ for $\alpha \in \mathcal{D}'$.
- Legendre constant: There exists a constant $L > 0$ such that

$$\left| z - \frac{P}{Q} \right| < L \cdot \frac{1}{|Q|^2}$$

implies that $\frac{P}{Q} = \frac{P_n}{Q_n}$ for some $n \geq 0 \rightarrow$?

Lakein's result

We use this to obtain a version of Lakein's result.

Theorem

Let $\alpha \in \mathcal{D}'$ and for $i = 1, 2$ set $A_i = \max\{\alpha_i, 1 - \alpha_i\}$. Then there is a constant

$$C_\alpha \geq 1 - \sqrt{\left(1 - \frac{A_1}{A_1^2 + A_2^2}\right)^2 + \left(1 - \frac{A_2}{A_1^2 + A_2^2}\right)^2}$$

such that

$$\left| z - \frac{P_n}{Q_n} \right| < \frac{1}{C_\alpha |Q_n|^2} \quad \text{for all } z \in U_\alpha,\ n \geq 1.$$

If $|A_1| = |A_2|$, then

$$C_\alpha = 1 - \sqrt{\left(1 - \frac{A_1}{A_1^2 + A_2^2}\right)^2 + \left(1 - \frac{A_2}{A_1^2 + A_2^2}\right)^2}.$$

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For $\alpha = (\frac{1}{2}, \frac{1}{2})$ we have $A_1 = A_2 = \frac{1}{2}$ and $C_\alpha = 1$.

For $\alpha \in \mathcal{D}'$ we get $\frac{1}{C_\alpha} < 1.00156$.

Legendre constant

Theorem

Let $\alpha \in \mathcal{D}' \cap \mathbb{Q}^2$. There exists an $L > 0$ such that for any $z \in U_\alpha$ and $(P, Q) \in \mathbb{Z}[i]^2$ the property

$$\left| z - \frac{P}{Q} \right| < L \cdot \frac{1}{|Q^2|}$$

implies that there is an $n \geq 0$ such that

$$\frac{P}{Q} = \frac{P_n(z)}{Q_n(z)}.$$

Natural extension

Theorem

Let $\alpha \in \mathcal{D}' \cap \mathbb{Q}^2$. Let $\{V_k\}$ denote the Markov partition of T_α. Write

$$V_k^* = \overline{\left\{ -\frac{Q_n}{Q_{n-1}} \; : \; z \in U_\alpha, \; T_\alpha^n(z) \in V_k, \; n \geq 1 \right\}}$$

and let $\hat{U}_\alpha = \bigcup_k V_k \times V_k^*$. Let

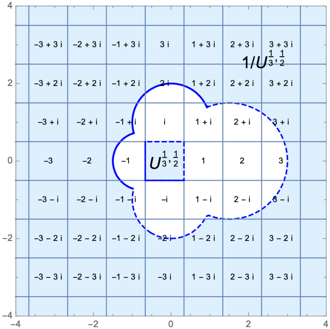
$$\hat{T}_\alpha(z, w) = \left(\frac{1}{z} - a_1^\alpha(z), \frac{1}{w} - a_1^\alpha(z) \right)$$

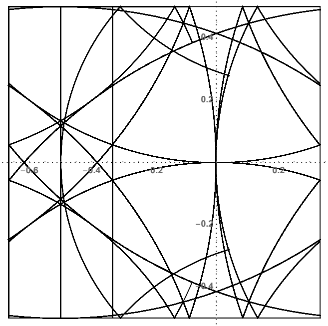
and let $\hat{\mu}_\alpha$ be the measure on $\hat{U}_\alpha$ with density $\hat{h}(z, w) = \frac{1}{|z-w|^4}$. Then $(\hat{U}_\alpha, \hat{\mu}_\alpha, \hat{T}_\alpha)$ is the natural extension of $(U, \mu_\alpha = \hat{\mu}_\alpha \circ \pi^{-1}, T)$.

So, we obtain an absolutely continuous invariant measure μ_α for T_α.

Non-increasing $|Q_n|$'s

The value $\alpha = (\frac{1}{3}, \frac{1}{2})$ was studied in the master thesis of Baasdam.





Non-increasing $|Q_n|$

For $\alpha = (\frac{1}{3}, \frac{1}{2})$ the sequence $(|Q_n|)$ is not necessarily increasing.

Example:

$$\frac{-3+i}{5} = \cfrac{1}{-1 + \cfrac{1}{-1+i}}.$$

This gives

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Proposition (Baasdam, K. and Thuswaldner)

- If $|Q_n| < |Q_{n-1}|$ for some $n \geq 2$, then $|Q_{n-2}| \leq |Q_{n-1}|$.
- $|\frac{Q_n}{Q_{n-1}}| \geq \sqrt{2} - 1$.

Not precisely a natural extension yet

A **natural extension** $(\hat{U}_\alpha, \hat{\mathcal{B}}_\alpha, \hat{\mu}_\alpha, \hat{T}_\alpha)$ of the system $(U_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ satisfies the following properties:

- There is a factor map $\pi : \hat{U}_\alpha \to U_\alpha$ preserving the measure $(\mu_\alpha = \hat{\mu}_\alpha \circ \pi^{-1})$ and the dynamics $(\pi \circ \hat{T}_\alpha = T_\alpha \circ \pi)$.
- $\hat{T}_\alpha$ is bijective $\hat{\mu}_\alpha$-a.e.
- $0 < \hat{\mu}_\alpha(\hat{U}_\alpha) < \infty$.
- $\bigvee_{k=0}^{\infty} \hat{T}_\alpha^k(\pi^{-1}\mathcal{B}_\alpha) = \hat{\mathcal{B}}_\alpha$.

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In the thesis it is shown that for $\alpha = (\frac{1}{3}, \frac{1}{2})$ the following hold.

- The factor map is ok.
- $\hat{T}_\alpha$ is surjective Lebesgue a.e.
- $\hat{T}_\alpha$ is injective Lebesgue a.e. if $\hat{\mu}_\alpha(\hat{U}_\alpha) < \infty$.

Observations

1. The first big issue to solve is whether the Lebesgue measure of $\hat{U}_\alpha$ is positive. This uses the fact that the $|Q_n|$ are increasing and the existence of a Legendre constant: there is a constant $L > 0$ such that for any $z \in U$ and coprime pair $(P, Q) \in \mathbb{Z}[i]^2$,

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2. Another ingredient in the proof that $\hat{\mu}(\hat{U}) > 0$ is that there is a constant $C_0 > 0$ such that for all k, l,

$$V^*_{k,l} \cap \{z \in \mathbb{C} : |z| > C_0\} = \{z \in \mathbb{C} : |z| > C_0\}.$$

We wonder whether this is still true for $\alpha \notin \mathcal{D}'$. Simulations seem to suggest that the structure of the $V_{k,l}$ becomes more complicated.

One can prove $\hat{\mu}_\alpha(\hat{U}_\alpha) > 0$ also with a less strong statement.