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q-Recursive Sequences

UNIVERSITÄT KLAGENFURT

Prelude: Binary Sum of Digits

- Binary expansion $n = \sum_{j \geq 0} \varepsilon_j 2^j$
- Sum of digits $s(n) = \sum_{j \geq 0} \varepsilon_j$

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for $n \geq 0$ with $s(0) = 0$.
In other words:

$$s(2n) = s(n),$$
$$s(2n+1) = s(n) + 1$$

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for $n \geq 1$ with $M(0) = M(1) = 0$.

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for $n \geq 1$ with $M(0) = M(1) = 0$.
In other words:

$$M(2n) = 2M(n) + 2n - 1,$$
$$M(2n+1) = M(n) + M(n+1) + 2n$$

for $n \geq 1$ with $M(0) = M(1) = 0$.

(Un-)bordered Factors

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  - exists non-empty word $v \neq w$
  - v is prefix and suffix of w
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bordered factor	border	length
00	0	2
11	1	2
010	0	3
101	1	3
1010	10	4
0110100110	0110	10

unbordered factor	length
ε	0
0	1
1	1
01	2
10	2
011	3
110	3
100	3
001	3

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bordered factor	border	length
$t[5..6] = 00$	0	2
$t[1..2] = 11$	1	2
$t[3..5] = 010$	0	3
$t[2..4] = 101$	1	3
$t[2..5] = 1010$	10	4
$t[0..9] = 0110100110$	0110	10

unbordered factor	length
ε	0
$t[0..0] = 0$	1
$t[1..1] = 1$	1
$t[0..1] = 01$	2
$t[2..3] = 10$	2
$t[0..2] = 011$	3
$t[1..3] = 110$	3
$t[4..6] = 100$	3
$t[5..7] = 001$	3

Thue–Morse Sequence

$$t = 01101001\,10010110\,10010110\,01101001\ldots$$

Number of Unbordered Factors

Theorem (Goč–Henshall–Shallit 2013)

exists unbordered factor of length n in Thue–Morse sequence $\iff$ $(n)_2 \notin 1(01^*0)^*10^*1$

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- number $f(n)$ of unbordered factors of length n in the Thue–Morse sequence

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(n)$	1	2	2	4	2	4	6	0	4	4	4	4	12	0	4	4

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Theorem (Goč–Mousavi–Shallit 2013)

- *inequality $f(n) \le n$ holds for all $n \ge 4$*
- *$f(n) = n$ infinitely often*
- $\limsup_{n\ge1} \frac{f(n)}{n} = 1$

Recurrence Relations

- number $f(n)$ of unbordered factors of length n in Thue–Morse sequence
- recurrence relations

$$
\begin{aligned}
f(4n) &= 2f(2n) && (n \geq 2)\\
f(4n+1) &= f(2n+1) && (n \geq 0)\\
f(8n+2) &= f(2n+1) + f(4n+3) && (n \geq 1)\\
f(8n+3) &= -f(2n+1) + f(4n+2) && (n \geq 2)\\
f(8n+6) &= -f(2n+1) + f(4n+2) + f(4n+3) && (n \geq 2)\\
f(8n+7) &= 2f(2n+1) + f(4n+3) && (n \geq 3)
\end{aligned}
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Theorem (Goč–Mousavi–Shallit 2013)

$f(n)$ satisfies recurrence relations above

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- $f(n)$ is a 2-recursive sequence

q-Recursive Sequences

Sequence x with

$$x(q^M n + r) = \sum_{\ell \le k \le u} c_{r,k}\, x(q^m n + k)$$

for all $n \ge n_0$ and $0 \le r < q^M$.

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q-recursive sequence

q-Regular Sequences

Vector-valued sequence $v\colon \mathbb{N}_0 \to \mathbb{C}^D$ with

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Regular sequences as inhomogeneities in the recurrence of the q-recursive sequence are allowed.

Minimality of Linear Representations

- Given regular sequence x, find matrices A_0, …, A_r, row vector u, column vector w and vector valued sequence v of minimal dimension D such that $x(n) = uv(n)$, $v(0) = w$, and $v(qn + r) = A_r v(n)$ for all $n \geq 0$ and $0 \leq r < q$.

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- These conditions are also sufficient (Schützenberger 1961; Berstel–Reutenauer 2011; H.–Krenn–Lipnik 202?).

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- Implemented in SageMath.