Forays Beyond Dendricity

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Numeration 2023, ULiège
Belgium, 21 May 2023

Université Paris Cité

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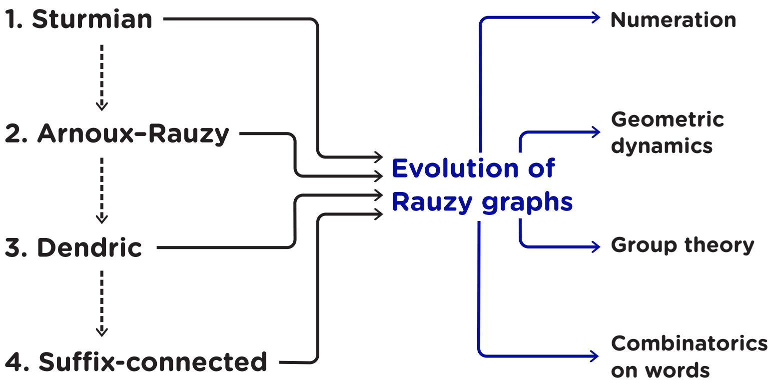
1. Sturmian
2. Arnoux-Rauzy
3. Dendric
4. Suffix-connected
Evolution of Rauzy graphs
Numeration
Geometric dynamics
Group theory
Combinatorics on words

Part 1

Sturmian

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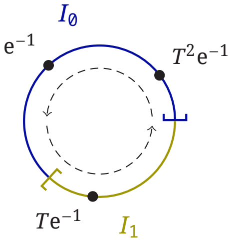

Standard Sturmian word of slope e^{-1}.

$\mathbf{s} = 010$

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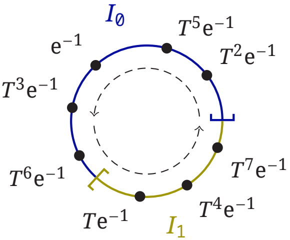

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$\mathbf{s} = 01001001\cdots$

Rauzy graphs

Rauzy graph: directed graph $\Gamma_n(\mathbf{w})$ with vertex set $\mathrm{fac}(\mathbf{w}) \cap A^n$.

There is an edge $u \xrightarrow{b} v$ when $ub = av \in \mathrm{fac}(w) \cap A^{n+1}$.

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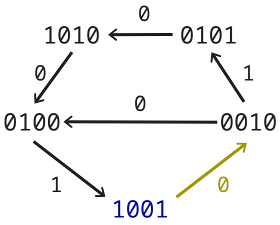

$\Gamma_4(\mathbf{s})$

$\mathbf{s} =$ 0100100101001001001010 $\cdots$

Special factors

- **Left special**. $au \in \mathrm{fac}(\mathbf{w})$ for *at least two* $a \in A$.
- **Right special**. $ub \in \mathrm{fac}(\mathbf{w})$ for *at least two* $b \in A$.
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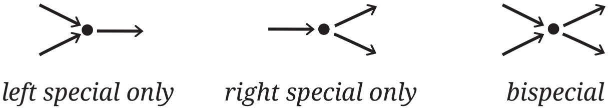

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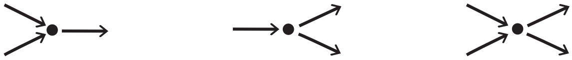
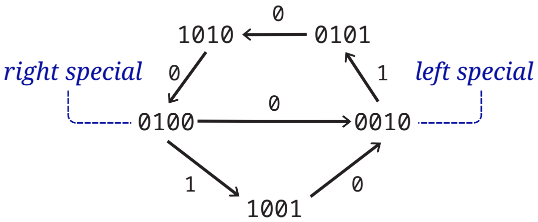


left special only *right special only* *bispecial*



Evolution of Rauzy graphs

Collapsing. Map $\Gamma_{n+1} \to \Gamma_n$ "erasing first letters", i.e. $ax \mapsto x$.

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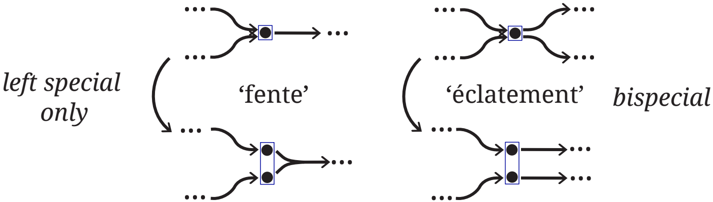

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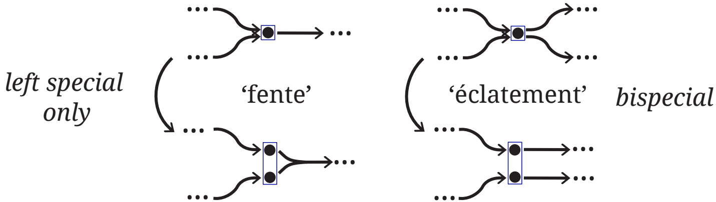

‘Fentes’ do not change the structure of the Rauzy graph.

Sturmian Rauzy graphs

Lemma. Sturmian words have exactly 1 left special factor G_n and one right special factor D_n for each $n \in \mathbb{N}$.

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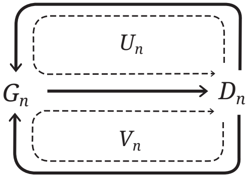
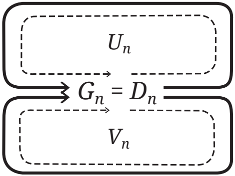
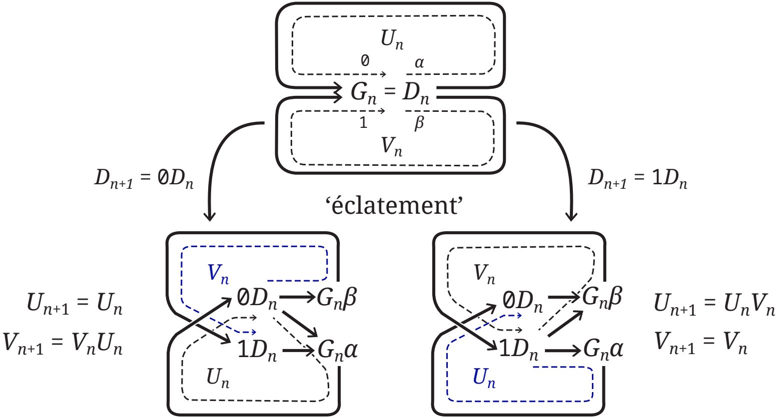


or

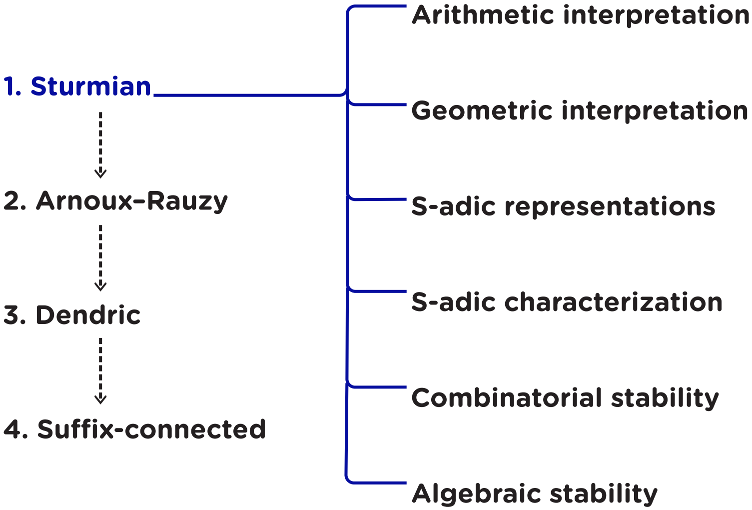


U_n and V_n := return words to D_n.

Sturmian ‘éclatements’



‘Éclatements’ $\rightleftarrows$ S-adic expansion.



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Part 2

Arnoux-Rauzy

Arnoux–Rauzy words

Arnoux & Rauzy, 1991. Generalized Sturmian words.

Arnoux–Rauzy word. Recurrent word $\mathbf{w} \in \{0, 1, 2\}^{\mathbb{N}}$ with for all n

- exactly one left special factor G_n, extended by 3 letters;
- exactly one right special word D_n, extended by 3 letters.

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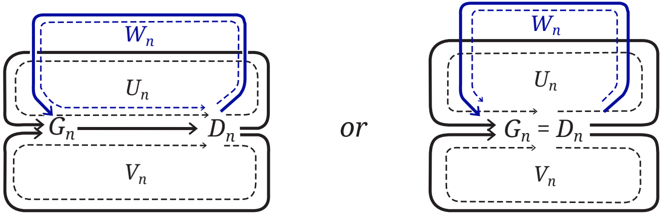
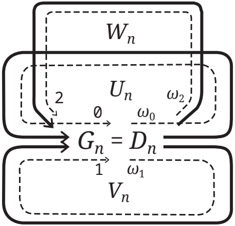


U_n, V_n and W_n := return words to D_n.

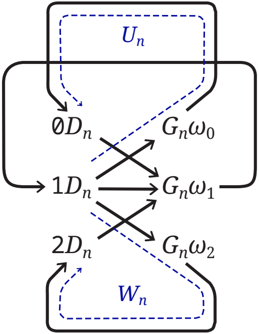
Arnoux–Rauzy ‘éclatements’



‘éclatement’

$D_{n+1} = 1D_n$

$$U_{n+1} = U_n V_n$$
$$V_{n+1} = V_n$$
$$W_{n+1} = W_n V_n$$



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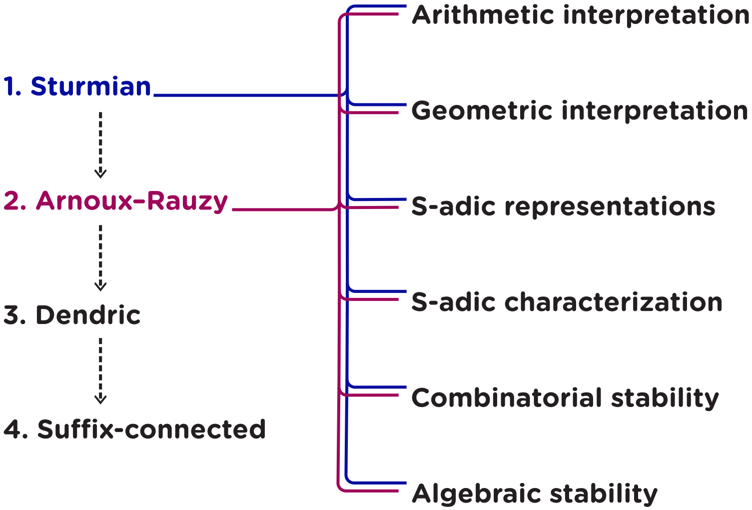
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Dendric

Extension graphs

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There is an edge $a — b$ if $aub \in \mathrm{fac}(\mathbf{w})$.

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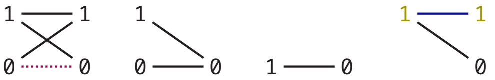


$\mathrm{E}_{\mathbf{s}}(0)$ $\mathrm{E}_{\mathbf{s}}(01)$ $\mathrm{E}_{\mathbf{s}}(001)$ $\mathrm{E}_{\mathbf{s}}(0010)$

$$\mathbf{s} = 0100100101001001001 0\cdots$$

$$000 \notin \mathrm{fac}(\mathbf{s})$$

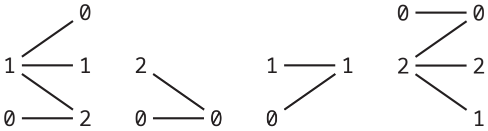
Dendric words

Berthé et al., 2015: generalized Arnoux–Rauzy.

Dendric word: all extension graphs are trees.

- Arnoux–Rauzy $\overset{\not\Leftarrow}{\Longrightarrow}$ Dendric.
- Dendric $\overset{\not\Leftarrow}{\Longrightarrow}$ Factor complexity $(k-1)n+1$.

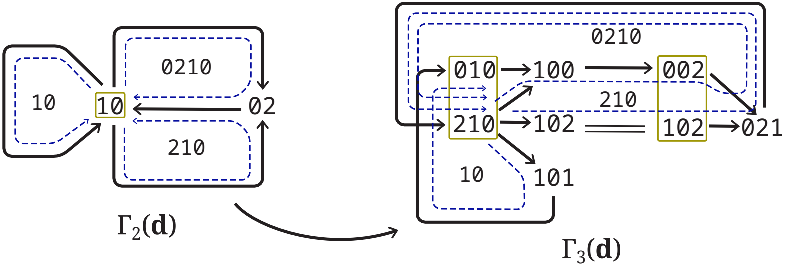
Let $\mathbf{d} = \mathtt{0210210100210}\cdots$ be the fixed point of the primitive morphism $\sigma\colon \mathtt{0} \mapsto \mathtt{0210}, \mathtt{1} \mapsto \mathtt{10}, \mathtt{2} \mapsto \mathtt{210}$.



$\mathrm{E}_{\mathbf{d}}(\mathtt{0})$ $\mathrm{E}_{\mathbf{d}}(\mathtt{1})$ $\mathrm{E}_{\mathbf{d}}(\mathtt{02})$ $\mathrm{E}_{\mathbf{d}}(\mathtt{10})$

Dendric 'éclatement'

'Éclatement' with rule $(U_{n+1}, V_{n+1}, W_{n+1}) = (U_n V_n, V_n, W_n)$.



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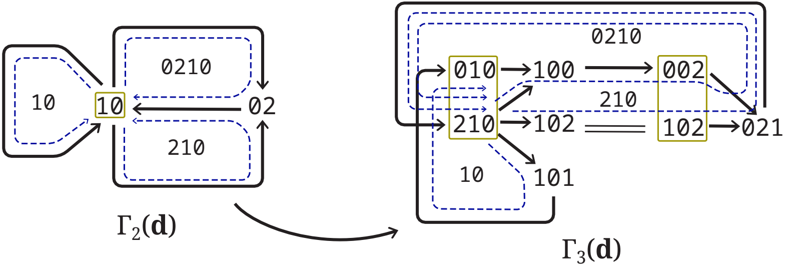

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Gheeraert & Leroy, 2022. S-adic characterization of dendricity.

Local ‘éclatement’ structure

Local principle. ‘Éclatements’ are locally determined by extension graphs.

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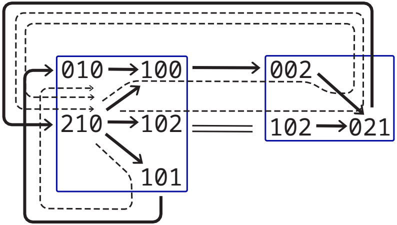
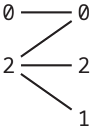
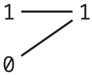




$E_{\mathbf{d}}(10)$



$E_{\mathbf{d}}(02)$

Rauzy groups

Fix an infinite word **w** on k letters.

Free group. $F_k :=$ the free group over k elements.

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Rauzy groups. $H_n(\mathbf{w})$:= subgroup of F_k generated by loops in $\Gamma_n(\mathbf{w})$ based at some fixed v_0 (independent up to conjugacy).

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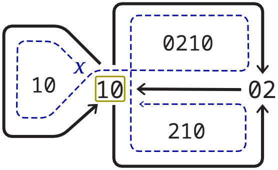

$\Gamma_2(\mathbf{d})$

$$x = (10)(0210)^{-1}(210) = 1$$

$$H_2(\mathbf{d}) = \langle 10, 0210, 210 \rangle = F_3$$

Return Theorem

Berthé et al., 2015. Let $\mathbf{w}$ be a recurrent dendric word on k letters.

1. For every n, $H_n(\mathbf{w}) = F_k$.
2. Sets of return words in $\mathbf{w}$ form bases of F_k.

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Combinatorial stability. Dendric words have constant number of return words (= size of the alphabet).

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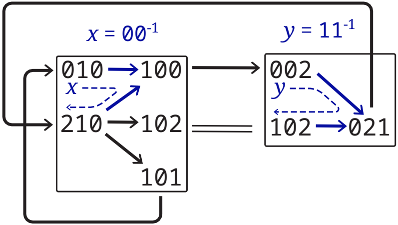
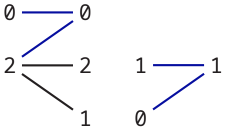
Algebraic stability. When all extension graphs are connected, collapsing maps $\implies$ Rauzy group *equalities*.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Gamma_3(\mathbf{w}) & \longrightarrow & \Gamma_2(\mathbf{w}) & \longrightarrow & \Gamma_1(\mathbf{w}) & \longrightarrow & \Gamma_0(\mathbf{w}) \\
 & & & & & & & & \downarrow \\
\cdots & = & H_3(\mathbf{w}) & = & H_2(\mathbf{w}) & = & H_1(\mathbf{w}) & = & H_0(\mathbf{w})
\end{array}$$

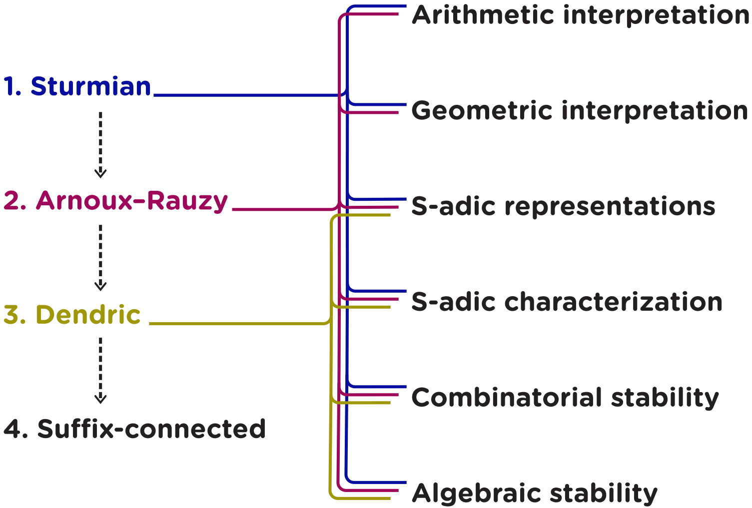
Algebraic stability

Path between left extensions $a, b \in \mathrm{E}_{\mathbf{w}}(u)$ *gives*
a trivial link $au \to bu$ *inside* $\Gamma_n(\mathbf{w})$.





$\mathrm{E}_{\mathbf{d}}(10)$ $\mathrm{E}_{\mathbf{d}}(02)$



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Part 4

Suffix-connected

A non-dendric example

G.-O., 2022. Let $\mathbf{c} = \texttt{0001000100010}\texttt{20} \cdots$ be the fixed point of the primitive morphism $\sigma\colon \texttt{0} \mapsto \texttt{0001}, \texttt{1} \mapsto \texttt{02}, \texttt{2} \mapsto \texttt{001}$.

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1. $H_n(\mathbf{c}) = F_3$ for all n.
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3. $\mathbf{c}$ has factor complexity $\neq 2n + 1$.

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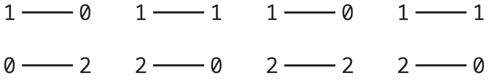
$$z_0 \coloneqq \texttt{001000100010}, \quad z_1 \coloneqq \sigma(z_0)\texttt{00}, \quad z_2 \coloneqq \sigma(z_1)\texttt{0}, \quad z_3 \coloneqq \sigma(z_2)\texttt{00}.$$



$\mathrm{E}_{\mathbf{c}}(z_0)$ $\mathrm{E}_{\mathbf{c}}(z_1)$ $\mathrm{E}_{\mathbf{c}}(z_2)$ $\mathrm{E}_{\mathbf{c}}(z_3)$

Suffix-connectedness

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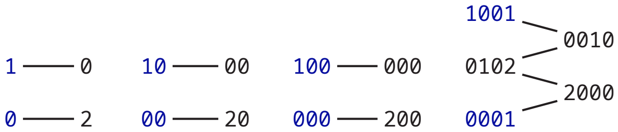

$E_{\mathbf{c}}^{(1)}(z_0^{(0)})$ $E_{\mathbf{c}}^{(2)}(z_0^{(1)})$ $E_{\mathbf{c}}^{(3)}(z_0^{(2)})$ $E_{\mathbf{c}}^{(4)}(z_0^{(3)})$

$$z_0^{(0)} = \texttt{001000100010}, \qquad z_0^{(1)} = \texttt{01000100010},$$
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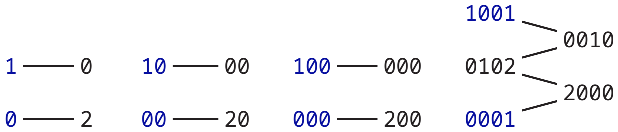

$\mathrm{E}_{\mathbf{c}}^{(1)}(z_0^{(0)})$ $\mathrm{E}_{\mathbf{c}}^{(2)}(z_0^{(1)})$ $\mathrm{E}_{\mathbf{c}}^{(3)}(z_0^{(2)})$ $\mathrm{E}_{\mathbf{c}}^{(4)}(z_0^{(3)})$

$$z_0^{(0)} = 001000100010, \qquad z_0^{(1)} = 01000100010,$$
$$z_0^{(2)} = 1000100010, \qquad z_0^{(3)} = 000100010.$$

Suffix-connected. Left extensions of every word $u \in \mathrm{fac}(\mathbf{w})$ are linked inside *some* suffix extension graph of u.

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Trivially linked. Path $u \to v$ inside $\Gamma_n(\mathbf{w})$ with trivial label in F_k.

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Lemma 2: Path in suffix extension graph $\leadsto$ trivially linked left extensions.

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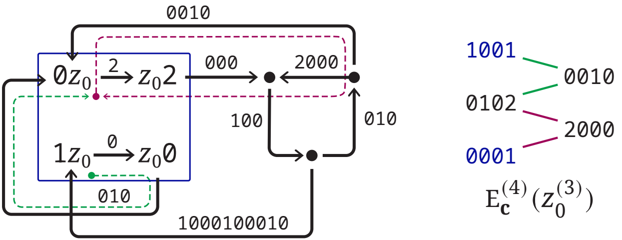
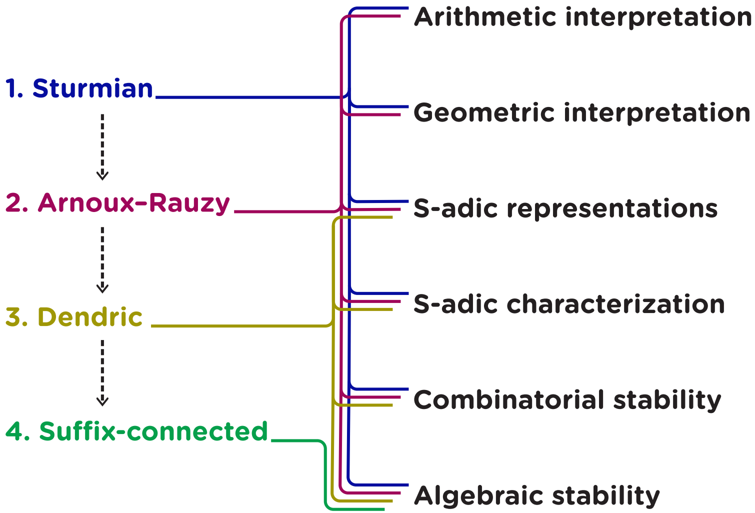


Stability theorem

G.-O., 2022. Let $\mathbf{w}$ be a uniformly recurrent suffix-connected word on k letters.

1. For every n, $H_n(\mathbf{w}) = F_k$.
2. Sets of return words in $\mathbf{w}$ generate F_k.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References

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