Numeration systems with a Delone spectrum

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Let $q > 1$ be a real number, and $D$ a finite set of non-negative real numbers with $0 \in D$. Write

$$X_D(q) := \left\{ \sum_{i=0}^{n} a_i q^i : a_i \in D, \ n = 0, 1, \ldots \right\}.$$

Let $0 = x_0 < x_1 < x_2 < \cdots$ be all the elements in $X_D(q)$.

**Definition.** $X_D(q)$ is said to be a Delone set in $(0, \infty)$ if $\exists \ c_1, c_2 > 0$ such that

$$c_1 \leq x_{n+1} - x_n \leq c_2, \quad n = 0, 1, \ldots.$$
We will consider the following question:

**Question A:** For which pair $(q, D)$ is the set $X_D(q)$ a Delone set in $(0, \infty)$?
Question A is closely related to a project proposed by Erdős, Joó and Komornik in the last 90’s. For $m \in \mathbb{N}$ and $q > 1$, set

$$X_m(q) = \left\{ \sum_{i=0}^{n} \epsilon_i q^i : \epsilon_i \in \{0, 1, \ldots, m\}, \ n = 0, 1, \ldots \right\}.$$  

Clearly $X_m(q) = X_D(q)$ with $D = \{0, 1, \ldots, m\}$. Rearrange the points of $X_m(q)$ into an increasing sequence:

$$0 = x_0(q, m) < x_1(q, m) < x_2(q, m) < \cdots.$$  

Denote

$$\ell_m(q) = \liminf_{n \to \infty} (x_{n+1}(q, m) - x_n(q, m)),$$

$$L_m(q) = \limsup_{n \to \infty} (x_{n+1}(q, m) - x_n(q, m)).$$

**Question by Erdős, Joó and Komornik:** When $\ell_m(q) = 0$? when $L_m(q) = 0$?.
Clearly, $X_m(q)$ is a Delone set $\iff 0 < \ell_m(q) \leq L_m(q) < \infty$.

(Drobot 1973) $\ell_m(q) = 0 \iff X_m(q) - X_m(q)$ is dense in $\mathbb{R}$.

A simple fact: $L_m(q) < \infty \iff q \leq m + 1$.
(Erdős-Joó-Komornik 1990)

It is well known that $\ell_m(q) > 0$ in the following two cases: $q$ is a Pisot number (Garsia 1962), or $q \geq m + 1$ (Erdős and Komornik 1998);

Recall that a **Pisot number** is an algebraic integer $> 1$ all of whose conjugates have modulus $< 1$. e.g. $(\sqrt{5} + 1)/2$ is a Pisot number, whose conjugate $= (1 - \sqrt{5})/2 \approx -0.618...$;
We include a brief proof for the fact that $\ell_m(q) > 0$ if $q$ is a Pisot number or $q \geq m + 1$.

First assume that $q$ is a Pisot number. Denote by $q_1, \ldots, q_d$ the algebraic conjugates of $q$. Then $\rho := \max_{1 \leq j \leq d} |q_j| < 1$. Let $P(x) = \sum_{i=0}^n \epsilon_i x^i$ be a polynomial with coefficients in $\{0, \pm 1, \ldots, \pm m\}$. Suppose that $P(q) \neq 0$. Then $P(q_j) \neq 0$ for $1 \leq j \leq d$. Hence $P(q) \prod_{j=1}^d P(q_j)$ is a non-zero integer. Therefore

$$|P(q)| \geq \prod_{j=1}^d \frac{1}{|P(q_j)|} \geq \left( \frac{1}{\sum_{i=0}^n m \rho^i} \right)^d > m^{-d}(1 - \rho)^d.$$ 

It follows that $\ell_m(q) \geq m^{-d}(1 - \rho)^d$. \hfill \qed
Next assume that $q \geq m + 1$. Then for any $n \in \mathbb{N}$,

$$q^n - \sum_{i=0}^{n-1} mq^i = \frac{q^n(q - 1 - m) + m}{q - 1} \geq \frac{(q - 1 - m) + m}{q - 1} = 1.$$ 

It follows that $|P(q)| \geq 1$ for any polynomial $P$ with degree $\geq 1$ and coefficients in $\{0, \pm 1, \ldots, \pm m\}$. Hence $\ell_m(q) \geq 1$. $\square$
We have seen that $\ell_m(q) > 0$ if $q$ is Pisot or $q \geq m + 1$. In 1998, Erdős, Joó and Komornik raised the following

**Question**: whether or not $\ell_m(q) = 0$ for any non-Pisot number $q \in (1, m + 1)$?
Previous partial results on $\ell_m(q)$ before 2013

- **(Drobot, 1973)** If $q \in (1, m + 1)$ does not satisfy an algebraic equation with coefficients $0, \pm 1, \ldots, \pm m$, then $\ell_m(q) = 0$.

- **(Bugeaud, 1996)** If $q$ is not a Pisot number, then $\ell_m(q) = 0$ when $m$ is sufficiently enough.

- **(Erdős and Komornik, 1998)** $\ell_m(q) = 0$ if $q$ is not a Pisot number and $m \geq \lceil q - q^{-1} \rceil + \lceil q - 1 \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

- **(Sidorov and Solomyak, 2011)** If $q \in (1, m + 1)$ and $q$ is not a Perron number, then $\ell_m(q) = 0$. Recall that an algebraic integer $q > 1$ is called a **Perron number** if each of its conjugates is less than $q$ in modulus.
Theorem (Akiyama-Komornik 2013, F. 2016)
\[ \ell_m(q) = 0 \iff q \text{ is a non-Pisot number in } (1, m + 1). \]

Remark: Let \( q \in (1, m + 1) \).

- **Akiyama-Komornik 2013:**
  \[ X_m(q) - X_m(q) \text{ has no finite accumulation pts } \iff q \text{ is Pisot}. \]

- **F. 2016:**
  \[ \ell_m(q) > 0 \iff X_m(q) - X_m(q) \text{ has no finite accumulation pts}. \]

Corollary (Akiyama-Komornik 2013, F. 2016)
\[ X_m(q) \text{ is a Delone set in } [0, \infty) \iff q \text{ is a Pisot number in } (1, m + 1). \]
Hence Question A (i.e. when $X_D(q)$ is Delone) has been well understood in the case when $D = \{0, 1, \ldots, m\}$.

Q: what can we say about the other cases that $D \neq \{0, 1, \ldots, m\}$?

Remark: For each $c > 0$, $X_{cD}(q) = cX_D(q)$. Hence

\[ X_D(q) \text{ is Delone } \iff X_{cD}(q) \text{ is Delone.} \]
Multiplying $D$ by a constant if necessary, we may assume that

$$0 = a_1 < a_2 < \ldots < a_m = 1 - q^{-1}$$

are the elements of $D$.

Consider the iterated function system

$$\Phi = \{\phi_i(x) = x/q + a_i\}_{i=1}^m.$$ 

Let $K$ be the attractor of $\Phi$, i.e., $K$ is the unique nonempty compact such that

$$K = \bigcup_{i=1}^m \phi_i(K).$$
Question A in the setting of IFS.

It is easy to see that $K \subset [0, 1]$ and $0, 1 \in K$.

By definition, $X_D(q)$ is Delone if and only if

- $K = [0, 1]$;
- $\Phi$ satisfies the **weak separation condition**, i.e. $0$ is not an accumulation point of $X_D(q) - X_D(q)$.

By [F. 2016], if $K = [0, 1]$ then

$\Phi$ satisfies the weak separation condition $\iff$ $\Phi$ satisfies the **finite type condition** (i.e. $X_D(q) - X_D(q)$ has no finite accumulation points).

**Question A**: Characterize $(q, D)$ so that the IFS $\Phi = \{x/q + a\}_{a \in D}$ satisfies the finite type condition, and has $[0, 1]$ as its attractor.
Let $q > 1$ and $D = \{a_i\}_{i=1}^m$ with

$$0 = a_1 < a_2 < \cdots < a_m = 1 - q^{-1}.$$ 

An easy fact:

$$K = [0, 1] \iff a_{i+1} - a_i \leq q^{-1} \text{ for all } i = 1, \ldots, m - 1.$$
Our results: a necessary condition

From now on we always assume that $q > 1$ and $D = \{a_i\}_{i=1}^m$ satisfies

\[0 = a_1 < a_2 < \cdots < a_m = 1 - q^{-1}\] and

\[a_{i+1} - a_i \leq q^{-1}\] for all $i = 1, \ldots, m - 1$.

Theorem

Suppose that $X_D(q)$ is a Delone set in $[0, \infty)$. Then $q$ is a Perron number, and $a_i \in \mathbb{Q}(q)$.

Recall that a Perron number is a positive algebraic integer $\lambda$ whose algebraic conjugates are strictly less than $\lambda$ in modulus.

Idea of the proof: build suitable substitutions by using the finite type condition of IFS, and to improve a result of [F. 2003].
A sufficient condition: If $q$ is Pisot and $a_i \in \mathbb{Q}(q)$ for all $i$, then $X_D(q)$ is Delone in $[0, \infty)$.

The proof is similar to that for the case when $D = \{0, 1, \ldots, m\}$.

Similar to the question of Erdős-Joó-Komornik, one may ask

Question: Under our general assumptions on $(q, D)$, do we always have the implication:

$$X_D(q) \text{ is Delone in } [0, \infty) \implies q \text{ is Pisot?}$$
Our results

Theorem

Suppose that $2 \leq \#D \leq 4$ and $X_D(q)$ is Delone in $[0, \infty)$. Then $q$ is Pisot.

For each $m \geq 5$, there exists $(q, D)$ with $\#D = m$ such that $q$ is not Pisot, but $X_D(q)$ is Delone in $[0, \infty)$. 

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Our results

**Theorem**

Suppose that $X_D(q)$ is Delone in $[0, \infty)$. If one of the following holds:

- $cD \subset \mathbb{Q}$ for some $c > 0$, or
- $a_{i+1} - a_i \leq 1/(2q)$ for all $i$,

then $q$ is Pisot.
Below we give an examples of \((q, D)\) with \(#D = 5\) such that \(q\) is not Pisot but \(X_D(q)\) is Delone.

### Example

Let \(q \approx 4.83442\) be the largest root of \(x^5 - 4x^4 - 3x^3 - 4x^2 - 4x - 4\). Let \(D = \{a_i\}_{i=1}^5\) be given by

\[
\begin{array}{c|c|c|c|c}
 a_1 & a_2 & a_3 & a_4 & a_5 \\
 0 & \frac{q^2-1}{q(q^2+1)} & \frac{q-1}{2q} & \frac{q^2-2q+1}{q^2+1} & \frac{q-1}{q} \\
\end{array}
\]

Then \(X_D(q)\) is Delone. In this example, \(q\) is non-Pisot and non-Salem.

The conjugates of \(q \approx 0.28501 \pm 0.97678i, -0.70222 \pm 0.55321i\)
Thank you for your attention !!!