Numeration systems with a Delone spectrum

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Introduction

Let $q > 1$ be a real number, and D a finite set of non-negative real numbers with $0 \in D$. Write

$$X_D(q) := \left\{ \sum_{i=0}^{n} a_i q^i : \ a_i \in D, \ n = 0, 1, \ldots \right\}.$$

Let $0 = x_0 < x_1 < x_2 < \cdots$ be all the elements in $X_D(q)$.

Defintion. $X_D(q)$ is said to be a Delone set in $(0, \infty)$ if $\exists \ c_1, c_2 > 0$ such that

$$c_1 \leq x_{n+1} - x_n \leq c_2, \qquad n = 0, 1, \ldots.$$

Main question in consideration

We will consider the following question:

Question A: For which pair (q, D) is the set $X_D(q)$ a Delone set in $(0, \infty)$?

Question A is closely related to a project proposed by Erdös, Joó and Komornik in the last 90's. For $m \in \mathbb{N}$ and $q > 1$, set

$$X_m(q) = \left\{ \sum_{i=0}^{n} \epsilon_i q^i : \ \epsilon_i \in \{0, 1, \dots, m\}, \ n = 0, 1, \dots \right\}.$$

Clearly $X_m(q) = X_D(q)$ with $D = \{0, 1, \dots, m\}$. Rearrange the points of $X_m(q)$ into an increasing sequence:

$$0 = x_0(q, m) < x_1(q, m) < x_2(q, m) < \cdots .$$

Denote

$$\ell_m(q) = \liminf_{n \to \infty} (x_{n+1}(q, m) - x_n(q, m)),$$
$$L_m(q) = \limsup_{n \to \infty} (x_{n+1}(q, m) - x_n(q, m)).$$

Question by Erdős, Joó and Komornik: When $\ell_m(q) = 0$? when $L_m(q) = 0$?.

Connection to a project by Erdős, Joó and Komornik

- Clearly, $X_m(q)$ is a Delone set $\Longleftrightarrow 0 < \ell_m(q) \le L_m(q) < \infty$.
- (Drobot 1973) $\ell_m(q) = 0 \Longleftrightarrow X_m(q) - X_m(q)$ is dense in $\mathbb{R}$.
- A simple fact: $L_m(q) < \infty \Longleftrightarrow q \le m+1$.
  (Erdős-Joó-Komornik 1990)
- It is well known that $\ell_m(q) > 0$ in the following two cases: q is a Pisot number (Garsia 1962), or $q \ge m+1$ (Erdős and Komornik 1998);
- Recall that a **Pisot number** is an algebraic integer > 1 all of whose conjugates have modulus < 1. e.g. $(\sqrt{5}+1)/2$ is a Pisot number, whose conjugate $= (1-\sqrt{5})/2 \approx -0.618...$;

We include a brief proof for the fact that $\ell_m(q) > 0$ if q is a Pisot number or $q \geq m+1$.

First assume that q is a Pisot number. Denote by $q_1, \ldots, q_d$ the algebraic conjugates of q. Then $\rho := \max_{1 \leq j \leq d} |q_j| < 1$. Let $P(x) = \sum_{i=0}^{n} \epsilon_i x^i$ be a polynomial with coefficients in $\{0, \pm 1, \ldots, \pm m\}$. Suppose that $P(q) \neq 0$. Then $P(q_j) \neq 0$ for $1 \leq j \leq d$. Hence $P(q) \prod_{j=1}^{d} P(q_j)$ is a non-zero integer. Therefore

$$|P(q)| \geq \prod_{j=1}^{d} \frac{1}{|P(q_j)|} \geq \left(\frac{1}{\sum_{i=0}^{n} m\rho^i} \right)^d > m^{-d}(1-\rho)^d.$$

It follows that $\ell_m(q) \geq m^{-d}(1-\rho)^d$. □

Next assume that $q \geq m+1$. Then for any $n \in \mathbb{N}$,

$$q^n - \sum_{i=0}^{n-1} mq^i = \frac{q^n(q-1-m)+m}{q-1} \geq \frac{(q-1-m)+m}{q-1} = 1.$$

It follows that $|P(q)| \geq 1$ for any polynomial P with degree ≥ 1 and coefficients in $\{0, \pm 1, \dots, \pm m\}$. Hence $\ell_m(q) \geq 1$. □

Connection to a project by Erdős, Joó and Komornik

We have seen that $\ell_m(q) > 0$ if q is Pisot or $q \geq m+1$. In 1998, Erdős, Joó and Komornik raised the following

Question: whether or not $\ell_m(q) = 0$ for any non-Pisot number $q \in (1, m+1)$?

Previous partial results on $\ell_m(q)$ before 2013

- (Drobot, 1973) If $q \in (1, m+1)$ does not satisfy an algebraic equation with coefficients $0, \pm 1, \ldots, \pm m$, then $\ell_m(q) = 0$.
- (Bugeaud, 1996) if q is not a Pisot number, then $\ell_m(q) = 0$ when m is sufficiently enough.
- (Erdös and Komornik, 1998) $\ell_m(q) = 0$ if q is not a Pisot number and $m \geq \lceil q - q^{-1} \rceil + \lceil q - 1 \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$.
- (Sidorov and Solomyak, 2011) if $q \in (1, m+1)$ and q is not a Perron number, then $\ell_m(q) = 0$. Recall that an algebraic integer $q > 1$ is called a **Perron number** if each of its conjugates is less than q in modulus.

Connection to a project by Erdős, Joó and Komornik

Theorem (Akiyama-Komornik 2013, F. 2016)

$\ell_m(q) = 0 \iff q$ *is a non-Pisot number in* $(1, m+1)$.

Remark: Let $q \in (1, m+1)$.

- Akiyama-Komornik 2013:
  $X_m(q) - X_m(q)$ has no finite accumulation pts $\iff q$ is Pisot.
- F. 2016:
  $\ell_m(q) > 0 \iff X_m(q) - X_m(q)$ has no finite accumulation pts.

Corollary (Akiyama-Komornik 2013, F. 2016)

$X_m(q)$ *is a Delone set in* $[0, \infty) \iff q$ *is a Pisot number in* $(1, m+1)$.

Targeted problem

Hence Question A (i.e. when $X_D(q)$ is Delone) has been well understood in the case when $D = \{0, 1, \ldots, m\}$.

Q: what can we say about the other cases that $D \neq \{0, 1, \ldots, m\}$?

Remark: For each $c > 0$, $X_{cD}(q) = cX_D(q)$. Hence

$$X_D(q) \text{ is Delone } \iff X_{cD}(q) \text{ is Delone.}$$

Question A in the setting of IFS.

- Multiplying D by a constant if necessary, we may assume that

$$0 = a_1 < a_2 < \ldots < a_m = 1 - q^{-1}$$

are the elements of D.

- Consider the iterated function system

$$\Phi = \{\phi_i(x) = x/q + a_i\}_{i=1}^{m}.$$

Let K be the attractor of Φ, i.e., K is the unique nonempty compact such that

$$K = \bigcup_{i=1}^{m} \phi_i(K).$$

Question A in the setting of IFS.

It is easy to see that $K \subset [0,1]$ and $0, 1 \in K$.

By definition, $X_D(q)$ is Delone if and only if

- $K = [0,1]$;
- Φ satisfies the weak separation condition, i.e. 0 is not an accumulation point of $X_D(q) - X_D(q)$.

By [F. 2016], if $K = [0,1]$ then
Φ satisfies the weak separation condition $\Longleftrightarrow$ Φ satisfies the finite type condition (i.e. $X_D(q) - X_D(q)$ has no finite accumulation points).

Question A: Charaterize (q, D) so that the IFS $\Phi = \{x/q + a\}_{a \in D}$ satisfies the finite type condition, and has $[0,1]$ as its attractor.

Let $q > 1$ and $D = \{a_i\}_{i=1}^m$ with

$$0 = a_1 < a_2 < \cdots < a_m = 1 - q^{-1}.$$

An easy fact:

$$K = [0, 1] \Longleftrightarrow a_{i+1} - a_i \leq q^{-1} \text{ for all } i = 1, \ldots, m-1.$$

Our results: a necessary condition

From now on we always assume that $q > 1$ and $D = \{a_i\}_{i=1}^m$ satisfies

$$0 = a_1 < a_2 < \cdots < a_m = 1 - q^{-1} \text{ and}$$
$$a_{i+1} - a_i \le q^{-1} \quad \text{for all } i = 1, \ldots, m-1.$$

Theorem

Suppose that $X_D(q)$ is a Delone set in $[0, \infty)$. Then q is a Perron number, and $a_i \in \mathbb{Q}(q)$.

Recall that a Perron number is a positive algebraic integer λ whose algebraic conjugates are strictly less than λ in modulus.

Idea of the proof: build suitable substitutions by using the finite type condition of IFS, and to improve a result of [F. 2003].

A sufficient condition: If q is Pisot and $a_i \in \mathbb{Q}(q)$ for all i, then $X_D(q)$ is Delone in $[0, \infty)$.

The proof is similar to that for the case when $D = \{0, 1, \dots, m\}$.

Similar to the question of Erdős-Joó-Komornik, one may ask

Question: Under our general assumptions on (q, D), do we always have the implication:

$$X_D(q) \text{ is Delone in } [0, \infty) \Longrightarrow q \text{ is Pisot?}$$

Our results

Theorem

- *Suppose that* $2 \leq \#D \leq 4$ *and* $X_D(q)$ *is Delone in* $[0, \infty)$. *Then* q *is Pisot.*
- *For each* $m \geq 5$, *there exists* (q, D) *with* $\#D = m$ *such that* q *is not Pisot, but* $X_D(q)$ *is Delone in* $[0, \infty)$.

Our results

Theorem

Suppose that $X_D(q)$ is Delone in $[0, \infty)$. If one of the following holds:

- $cD \subset \mathbb{Q}$ *for some* $c > 0$, *or*
- $a_{i+1} - a_i \leq 1/(2q)$ *for all* i,

then q *is Pisot.*

Our example

Below we give an examples of (q, D) with $\#D = 5$ such that q is not Pisot but $X_D(q)$ is Delone.

Example

Let $q \approx 4.83442$ be the largest root of $x^5 - 4x^4 - 3x^3 - 4x^2 - 4x - 4$. Let $D = \{a_i\}_{i=1}^5$ be given by

a_1	a_2	a_3	a_4	a_5
0	$\frac{q^2-1}{q(q^2+1)}$	$\frac{q-1}{2q}$	$\frac{q^2-2q+1}{q^2+1}$	$\frac{q-1}{q}$

Then $X_D(q)$ is Delone. In this example, q is non-Pisot and non-Salem.

The conjugates of $q \approx 0.28501 \pm 0.97678i$, $-0.70222 \pm 0.55321i$

Thank you for your attention !!!